

# The Binomial Theorem

### Patterns, coefficients and generalized expansions

Narrative explanations • Worked examples • SL + HL material

**Focused chapter scope.** Pascal's Triangle and combinations; the finite binomial theorem; targeted powers and constant terms; parameter recovery; products of expansions; GDC strategy; IB-style past-paper patterns; a new sequences–binomial synthesis; and HL generalized binomial expansions, validity and approximation.

**Design of this revised edition.** These notes are written like a guided lesson rather than a formula sheet. The route is deliberately verbal before it is symbolic: locate the power, identify the term, evaluate the coefficient, and then interpret the result. This edition adds four further IB-style past-paper adaptations and two extra HL worked examples.

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**How to use the chapter.** Read the worked examples as conversations with the mathematics. For a targeted coefficient, do not expand everything: decide which selection creates the requested power. For HL infinite expansions, factor first, state the validity interval, and only then use approximation.

## 1 Before the theorem: what are we actually doing?

When students first meet the binomial theorem, the formula can look much more complicated than the idea behind it. So before we write any summation sign, let us begin with the action that creates an expansion.

Suppose we write

$$(a + b)^4 = (a + b)(a + b)(a + b)(a + b).$$

To create one term in the final expansion, we make *one choice from each bracket*. From every bracket we choose either  $a$  or  $b$ . If, for example, we want a term containing  $a^3b$ , then exactly one of the four brackets must supply a  $b$ , while the other three supply  $a$ .

Now comes the important question: *how many different ways can that happen?* The  $b$  could have come from the first bracket, the second, the third or the fourth. There are four routes, so the coefficient of  $a^3b$  is 4.

That is the story behind binomial coefficients: **the coefficient counts how many different selections create the same algebraic term.**

A **term** is the whole object, for example  $60x^3$ .

The **coefficient** is only the numerical multiplier, so the coefficient of  $x^3$  in  $60x^3$  is 60.

The **power** or **exponent** tells us which term we are looking for. In  $60x^3$ , the power of  $x$  is 3.

Many binomial mistakes happen because a student finds the right power but then reports the whole term when the question asked only for the coefficient, or vice versa. We will keep asking which of these three objects the question wants.

### 1.1 Pascal's Triangle: the coefficients before the formula

Pascal's Triangle is not simply a list of rows. It is a triangle built by one repeated rule: every interior number is the sum of the two numbers immediately above it. The outside edges are always 1.

|       |  |  |  |   |   |    |    |    |    |   |   |
|-------|--|--|--|---|---|----|----|----|----|---|---|
| row 0 |  |  |  | 1 |   |    |    |    |    |   |   |
| row 1 |  |  |  | 1 | 1 |    |    |    |    |   |   |
| row 2 |  |  |  | 1 | 2 | 1  |    |    |    |   |   |
| row 3 |  |  |  | 1 | 3 | 3  | 1  |    |    |   |   |
| row 4 |  |  |  | 1 | 4 | 6  | 4  | 1  |    |   |   |
| row 5 |  |  |  | 1 | 5 | 10 | 10 | 5  | 1  |   |   |
| row 6 |  |  |  | 1 | 6 | 15 | 20 | 15 | 6  | 1 |   |
| row 7 |  |  |  | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

The highlighted 10 is created by the two numbers above it:  $4 + 6 = 10$ . The same rule builds every interior entry. Row  $n$  contains the coefficients  $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ .

Now let us connect the triangle to algebra. Row 5 gives the coefficients of  $(a + b)^5$ :

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Read this from left to right. The power of  $a$  begins at 5 and decreases by 1 each time. At the same time, the power of  $b$  begins at 0 and increases by 1 each time. The total power in every term remains 5.

In a finite binomial expansion, one exponent goes down while the other goes up. If your exponents do not add to the original power  $n$ , something has gone wrong.

### Worked Example 1 - the counting story hidden inside algebra

Ten students volunteer and three must be chosen for a presentation team. How many teams are possible, and where does the same number appear in  $(a + b)^{10}$ ?

#### Solution - think first, calculate second

We are choosing 3 students from 10, and the order does not matter. The number of possible teams is therefore

$$\binom{10}{3} = 120.$$

Now let us translate that same choice into the expansion of  $(a + b)^{10}$ . A term containing exactly three copies of  $b$  must choose  $b$  from three of the ten brackets and  $a$  from the remaining seven. There are again  $\binom{10}{3} = 120$  ways to decide which three brackets supply  $b$ .

So the corresponding term is

$$\binom{10}{3} a^7 b^3 = 120 a^7 b^3.$$

The number 120 is not appearing by magic: it is literally counting the different selection routes that all collapse to the same term  $a^7 b^3$ .

## 2 The finite binomial theorem

We are now ready to compress the story into a formula. The formula is useful because it allows us to jump directly to any term without expanding every bracket.

For a non-negative integer  $n$ ,

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r.$$

The  $(r + 1)$ th term is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r.$$

Let us unpack this slowly. The letter  $r$  counts how many times the *second* part,  $b$ , has been selected. If  $r = 0$ , we select  $b$  zero times, so we get the first term. If  $r = 1$ , we select  $b$  once, so we get the second term. Therefore  $r = 3$  corresponds to the *fourth* term, not the third.

As  $r$  increases by 1, the power of  $b$  increases by 1, while the power of  $a$  decreases by 1. This is exactly the movement we saw in Pascal's Triangle.

$r$  counts second-term selections. The term number is  $r + 1$ . Thus  $r = 0$  gives  $T_1$ ,  $r = 1$  gives  $T_2$ , and  $r = 4$  gives  $T_5$ .

**Worked Example 2 - signs belong to the selected term**

Expand  $(2 - 3x)^6$  as far as the term in  $x^3$ .

**Solution**

Before writing anything, notice that the second part of the binomial is  $-3x$ , not merely  $3x$ . We must carry the negative sign with the term, because powers of  $-3x$  are what create the alternating signs in the expansion.

The power of  $x$  comes entirely from  $(-3x)^r$ . To reach  $x^3$ , we only need  $r = 0, 1, 2, 3$ . There is no reason to continue further.

$$\begin{aligned}(2 - 3x)^6 &= 2^6 + \binom{6}{1} 2^5(-3x) + \binom{6}{2} 2^4(-3x)^2 \\ &\quad + \binom{6}{3} 2^3(-3x)^3 + \dots\end{aligned}$$

Now we evaluate one term at a time:

$$(2 - 3x)^6 = 64 - 576x + 2160x^2 - 4320x^3 + \dots$$

Notice the workflow: first we decided how far we needed to go; only afterwards did we perform the arithmetic. Keeping  $(-3x)$  in brackets also protects the signs.

**Worked Example 3 - reading the general term**

Write the general term of

$$\left(3x^2 - \frac{2}{x}\right)^9$$

and simplify the power of  $x$ .

**Solution**

Here the first choice is  $3x^2$  and the second choice is  $-2/x$ . Therefore

$$T_{r+1} = \binom{9}{r} (3x^2)^{9-r} \left(-\frac{2}{x}\right)^r.$$

At this stage, the most important part is not the numerical coefficient. What matters is the exponent of  $x$ , because that exponent will tell us which value of  $r$  produces any requested power later.

The first factor contributes  $x^{2(9-r)}$ . The second contributes  $x^{-r}$ . Multiplying powers means adding exponents:

$$x^{2(9-r)} x^{-r} = x^{18-3r}.$$

So we may write

$$T_{r+1} = \binom{9}{r} 3^{9-r} (-2)^r x^{18-3r}.$$

The expression  $18 - 3r$  is the “power map” for the whole expansion. Once we have it, coefficient questions become much more controlled.

### 3 Targeting one power: Method A

Most IB binomial questions do not ask for the whole expansion. They ask for one coefficient, one term, or the term independent of  $x$ . In such questions the key skill is **locating the correct value of  $r$  before**

touching the numerical coefficient.

1. Write  $T_{r+1}$ .
2. Simplify only the power of  $x$ .
3. Set that exponent equal to the requested power.
4. Solve for  $r$  and check that it is an integer in  $0 \leq r \leq n$ .
5. Only now evaluate the numerical coefficient or complete term.

Do not evaluate  $\binom{n}{r}$  while  $r$  is still unknown. The coefficient is meaningless until you have proved which term you need.

#### Worked Example 4 - coefficient and constant term

In the expansion of

$$\left(x^2 + \frac{4}{x}\right)^{12},$$

find (a) the coefficient of  $x^6$ , and (b) the constant term.

#### Solution

**First build one general term that can answer both parts.** The first choice is  $x^2$  and the second is  $4/x$ , so

$$T_{r+1} = \binom{12}{r} (x^2)^{12-r} \left(\frac{4}{x}\right)^r = \binom{12}{r} 4^r x^{24-3r}.$$

Why is the exponent  $24 - 3r$ ? If we chose  $x^2$  from all 12 brackets we would start at power 24. Every time we replace one  $x^2$  by  $x^{-1}$ , the exponent drops by 3. The algebra and the verbal story agree.

**(a) Locate the  $x^6$  term.** We want the exponent in the general term to be 6, so

$$24 - 3r = 6.$$

This gives  $r = 6$ . Pause here:  $r = 6$  means that six of the twelve brackets contribute  $4/x$ , and the other six contribute  $x^2$ . We have now located the term.

**Now evaluate the coefficient.**

$$\binom{12}{6} 4^6 = 3\,784\,704.$$

Therefore the coefficient of  $x^6$  is

$$\boxed{3\,784\,704}.$$

**(b) What does “constant” mean?** A constant is  $x^0$ , so now we set the exponent equal to 0:

$$24 - 3r = 0 \quad \Rightarrow \quad r = 8.$$

Thus the constant term is

$$\binom{12}{8} 4^8 = 32\,440\,320.$$

Notice that we used the same general term twice. This is why setting up the structure carefully at the beginning saves work later.

**Worked Example 5 - a negative target power**

Find the coefficient of  $x^{-2}$  in

$$\left(x + \frac{2}{x^2}\right)^7.$$

**Solution**

A negative target power does not require a new theorem. We use exactly the same logic. The general term is

$$T_{r+1} = \binom{7}{r} x^{7-r} (2x^{-2})^r = \binom{7}{r} 2^r x^{7-3r}.$$

We are looking for  $x^{-2}$ , so the exponent must be  $-2$ :

$$7 - 3r = -2 \quad \Rightarrow \quad r = 3.$$

Now that the term has been located, we evaluate its coefficient:

$$\binom{7}{3} 2^3 = 35 \cdot 8 = \boxed{280}.$$

A useful mental check is that starting from  $x^7$ , three switches from  $x$  to  $x^{-2}$  lower the power by 3 each time:  $7 \rightarrow 4 \rightarrow 1 \rightarrow -2$ .

**4 Targeting one power: Method B, the power ledger**

Method A is universal, but sometimes the full general term feels heavier than necessary. Method B keeps the same mathematics but turns it into a bookkeeping story.

Suppose the two choices contribute powers  $x^u$  and  $x^v$ . If every bracket initially contributes  $x^u$ , the starting exponent is  $nu$ . Every time we switch one bracket from  $x^u$  to  $x^v$ , the total exponent changes by  $v - u$ .

**Start exponent  $\rightarrow$  change per switch  $\rightarrow$  target exponent.**

Return to

$$\left(x^2 + \frac{4}{x}\right)^{12}.$$

If all twelve brackets supply  $x^2$ , the starting power is 24. A switch from  $x^2$  to  $x^{-1}$  reduces the exponent by 3. To reach  $x^6$  we need a total drop of 18, which means  $18/3 = 6$  switches. Therefore  $r = 6$  immediately.

This is not a different theorem. It is simply a faster way to solve the exponent equation  $24 - 3r = 6$ .

Use the power ledger when the two powers are easy to compare mentally, for example  $x^3$  versus  $x^{-1}$  or  $x^2$  versus  $x^{-2}$ . Use Method A when several variables or complicated algebra make the general term clearer.

**5 Recovering unknown parameters after the power is fixed**

A common IB pattern is to tell you the value of a coefficient and ask you to recover an unknown parameter. The order of operations matters: **first use the power to determine  $r$ ; then use the coefficient to determine the parameter.**

**Worked Example 6 - recover a parameter**

The coefficient of  $x^{11}$  in

$$\left(x^2 + \frac{1}{ax}\right)^{10}$$

is 15. Find  $a$ .

**Solution**

Let us resist the temptation to use the number 15 immediately. We do not yet know which term produces  $x^{11}$ .

The general power of  $x$  is

$$2(10 - r) - r = 20 - 3r.$$

To obtain  $x^{11}$ ,

$$20 - 3r = 11 \quad \Rightarrow \quad r = 3.$$

Now the location is fixed. The coefficient in that term is

$$\binom{10}{3} a^{-3}.$$

The question tells us that this equals 15, so only now do we form the coefficient equation:

$$\binom{10}{3} a^{-3} = 15.$$

Since  $\binom{10}{3} = 120$ ,

$$\frac{120}{a^3} = 15 \quad \Rightarrow \quad a^3 = 8 \quad \Rightarrow \quad \boxed{a = 2}.$$

The story is therefore: power information located the term; coefficient information recovered the parameter.

**Worked Example 7 - two variables locate the term for us**

In

$$\left(y - \frac{a}{x}\right)^8,$$

the coefficient of  $\frac{y^2}{x^6}$  is 1792. Find all real  $a$ .

**Solution**

Here the variables themselves tell us how many selections must come from each part. To obtain  $y^2$ , exactly two of the eight brackets must supply  $y$ . Therefore the other six brackets must supply  $-a/x$ . Those six selections automatically create  $x^{-6}$ , which matches the required denominator  $x^6$ . So there is no need to solve a separate exponent equation. The relevant term has  $r = 6$ , and its coefficient is

$$\binom{8}{6} (-a)^6.$$

Because the sixth power removes the negative sign,

$$28a^6 = 1792.$$

Hence

$$a^6 = 64,$$

and the real solutions are

$$a = \pm 2.$$

The lesson here is that sometimes a variable pattern can locate the term faster than a formal exponent calculation.

### Worked Example 8 - term number information

The fourth term in the expansion of  $(ax + \sqrt{2})^{10}$  is  $30x^7$ . Find  $a$ .

#### Solution

The phrase “fourth term” gives us  $r$  immediately. Since term number is  $r + 1$ , the fourth term corresponds to  $r = 3$ .

Therefore

$$T_4 = \binom{10}{3} (ax)^7 (\sqrt{2})^3.$$

Now let us simplify the non- $x$  part carefully:

$$\binom{10}{3} = 120, \quad (\sqrt{2})^3 = 2\sqrt{2}.$$

Thus

$$T_4 = 240\sqrt{2} a^7 x^7.$$

The complete term is given as  $30x^7$ . The  $x^7$  already matches, so the coefficients must be equal:

$$240\sqrt{2} a^7 = 30.$$

Hence

$$a^7 = \frac{1}{8\sqrt{2}} = 2^{-7/2},$$

so

$$a = 2^{-1/2} = \frac{1}{\sqrt{2}}.$$

## 6 Coefficient relationships: turn words into equations

IB questions often hide the binomial theorem inside a sentence: “the coefficient of  $x^4$  is four times the coefficient of  $x^3$ ”, or “the coefficient of  $x^2$  plus 2 equals the square of the coefficient of  $x$ ”. The binomial work is only half the problem. The other half is translating the English sentence correctly.

### Worked Example 9 - a nonlinear coefficient relationship

In  $(1 + ax)^4$ , let  $A_1$  be the coefficient of  $x$  and  $A_2$  the coefficient of  $x^2$ . Given  $A_2 + 2 = A_1^2$ , find all real  $a$ .

#### Solution

First, identify the two coefficients separately. We do not need the whole expansion.

$$A_1 = \binom{4}{1} a = 4a, \quad A_2 = \binom{4}{2} a^2 = 6a^2.$$

Now translate the sentence exactly as written:

$$6a^2 + 2 = (4a)^2.$$

Therefore

$$6a^2 + 2 = 16a^2 \Rightarrow 10a^2 = 2 \Rightarrow a^2 = \frac{1}{5}.$$

Because the question asks for all real values and gives no sign restriction,

$$a = \pm \frac{1}{\sqrt{5}}.$$

Do not throw away the negative solution unless the question tells you  $a > 0$ .

### Worked Example 10 - compare neighbouring coefficients

In  $(1 + ax)^8$ , where  $a \neq 0$ , the coefficient of  $x^4$  is four times the coefficient of  $x^3$ . Find  $a$ .

#### Solution

The coefficient of  $x^4$  is  $\binom{8}{4}a^4$ , while the coefficient of  $x^3$  is  $\binom{8}{3}a^3$ . The sentence therefore becomes

$$\binom{8}{4}a^4 = 4\binom{8}{3}a^3.$$

Now use the information  $a \neq 0$ . It allows us to divide both sides by  $a^3$  without losing a possible solution:

$$70a = 4(56) = 224.$$

Hence

$$a = \frac{16}{5}.$$

The efficient route was to compare only the two coefficients mentioned in the question. Expanding eight terms would add work without adding understanding.

### Worked Example 11 - recover both $n$ and $a$

In  $(1 + ax)^n$ , where  $a > 0$  and  $n$  is a positive integer, the coefficient of  $x^2$  is 112 and the coefficient of  $x^3$  is 448. Find  $a$  and  $n$ .

#### Solution

The two pieces of information give two equations:

$$\binom{n}{2}a^2 = 112, \quad \binom{n}{3}a^3 = 448.$$

A direct substitution is possible, but there is a cleaner idea: *divide the equations*. This removes most of the common factorial structure and one power of  $a$ .

$$\frac{\binom{n}{3}a^3}{\binom{n}{2}a^2} = \frac{448}{112} = 4.$$

Now

$$\frac{\binom{n}{3}}{\binom{n}{2}} = \frac{n-2}{3},$$

so

$$\frac{n-2}{3}a = 4.$$

Therefore

$$a = \frac{12}{n-2}.$$

Substitute this into  $\binom{n}{2}a^2 = 112$ . The positive-integer solution is  $n = 8$ , and then

$$a = \frac{12}{6} = 2.$$

Thus

$$\boxed{n = 8, \quad a = 2}.$$

The strategic point is more important than the arithmetic: when two neighbouring binomial coefficients appear, taking a ratio often simplifies the problem dramatically.

### Worked Example 12 - coefficient data can determine the exponent

The coefficient of  $x^2$  in  $(1+x)^n$  is 45. Find the positive integer  $n$ .

#### Solution

In  $(1+x)^n$ , the coefficient of  $x^2$  is simply  $\binom{n}{2}$ . Therefore

$$\binom{n}{2} = 45.$$

Writing the combination explicitly,

$$\frac{n(n-1)}{2} = 45.$$

So

$$n^2 - n - 90 = 0 = (n-10)(n+9).$$

The problem specifies a positive integer, so

$$\boxed{n = 10}.$$

## 7 Products of expansions: powers must add

When two expansions are multiplied, the coefficient of  $x^k$  does not usually come from one term. It comes from *pairs* of powers whose exponents add to  $k$ .

If

$$A(x) = \sum_i a_i x^i, \quad B(x) = \sum_j b_j x^j,$$

then the coefficient of  $x^k$  in  $A(x)B(x)$  is obtained by adding  $a_i b_j$  over all pairs satisfying

$$i + j = k.$$

The practical message is simple: **list the possible power pairs first.**

**Worked Example 13 - a product with negative powers**

Find the coefficient of  $x^{-1}$  in

$$(x+2)^4 \left( \frac{1}{x} - 3x \right)^5.$$

**Solution**

The first factor can contribute powers  $x^0, x^1, x^2, x^3, x^4$ .

In the second factor, suppose we choose  $-3x$  exactly  $r$  times. The remaining  $5-r$  selections come from  $x^{-1}$ . Its exponent is therefore

$$-(5-r) + r = -5 + 2r.$$

If the first factor contributes  $x^i$ , the total exponent is

$$i + (-5 + 2r).$$

We want  $x^{-1}$ , so

$$i - 5 + 2r = -1 \iff i + 2r = 4.$$

Now we have reduced a complicated product to a short integer search. The possible pairs are

$$(i, r) = (4, 0), (2, 1), (0, 2).$$

Only these three combinations can contribute to  $x^{-1}$ . Evaluating their coefficients gives

$$1 - 360 + 1440 = \boxed{1081}.$$

The important step was not the arithmetic. It was proving that there are exactly three contributing power pairs.

**Worked Example 14 - simplify before expanding**

Find the coefficient of  $x^8$  in  $(1+x)^{10}(1-x)^{10}$ .

**Solution**

Before expanding anything, look for structure. The two factors have the same exponent, so

$$(1+x)^{10}(1-x)^{10} = ((1+x)(1-x))^{10} = (1-x^2)^{10}.$$

Now the problem is much smaller. To obtain  $x^8$ , we need  $(x^2)^4$ . Therefore the relevant term has  $r = 4$ , and because  $(-x^2)^4$  is positive, the coefficient is

$$\binom{10}{4} = \boxed{210}.$$

Always simplify the algebraic structure before launching into a long convolution.

## 8 GDC workshop: technology after the mathematics

A GDC is excellent for evaluating large combinations and for searching integer values once the mathematical equation is established. It should not be used to guess which term is required.

**You decide:** which term, which  $r$ , which coefficient equation, and what restrictions apply.

**The GDC can help with:**  $nCr$  values, tables of integer  $n$ , solving a polynomial equation, or checking a final numerical approximation.

### Worked Example 15 - integer search only after the equation is known

Find the positive integer  $n$  satisfying

$$\binom{n}{4} = 4845.$$

#### Solution and GDC route

The mathematics has already reduced the question to an integer equation. We are not asking the calculator to discover a binomial pattern; we are asking it to search values of a known function. A short table gives

$$\binom{20}{4} = 4845,$$

so

$$\boxed{n = 20}.$$

**Casio fx-CG50:** use a table with  $Y_1 = X nCr 4$  over a short integer range and locate 4845.

**TI-Nspire CX/CX II:** create consecutive  $n$ -values in Lists & Spreadsheet and evaluate  $nCr(n, 4)$  in the next column, or test a short integer range on a Calculator page.

## 9 HL MATERIAL: fractional and negative indices

### HL MATERIAL: Why the theorem changes

When the exponent is a non-integer or a negative integer, the expansion generally does not stop after finitely many terms. Instead, near  $x = 0$ , we use the generalized binomial series

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

For the standard form used in IB questions, the series is valid when  $|x| < 1$ . If the small quantity is  $u(x)$  rather than  $x$ , the condition becomes  $|u(x)| < 1$ .

The most important difference is conceptual. For a finite expansion,  $r$  eventually reaches  $n$  and the process stops. For a generalized expansion, the coefficient pattern continues indefinitely. So at HL we must think about two extra questions: *How many terms do we actually need?* and *for which values of  $x$  is the series valid?*

### HL MATERIAL: The six-step routine

1. Rewrite the expression as  $A(1+u)^\alpha$ . Factor constants first.
2. Decide the highest power needed and write only enough generalized-binomial terms to reach it.
3. Substitute  $u$  carefully, including its sign.
4. Simplify term by term; do not expand beyond the requested degree.
5. State  $|u| < 1$ . For products, intersect all validity conditions.
6. For approximation, rewrite the target so that  $u$  is genuinely small.

**HL Worked Example 16 - a fractional exponent**

Expand  $(1 + 2x)^{1/2}$  up to and including  $x^3$ , and state the range of validity.

**HL Solution**

Here  $\alpha = 1/2$  and the small quantity is  $u = 2x$ . Start with the generic cubic form:

$$(1 + u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 + \dots$$

Now substitute  $u = 2x$  only after the coefficient pattern is clear:

$$(1 + 2x)^{1/2} = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots$$

For validity, the generalized binomial series requires

$$|u| < 1.$$

Since  $u = 2x$ ,

$$|2x| < 1 \quad \Rightarrow \quad \boxed{|x| < \frac{1}{2}}.$$

**HL Worked Example 17 - factor first for a negative exponent**

Expand  $(2 - 3x)^{-2}$  up to  $x^3$  and state the range of validity.

**HL Solution**

The generalized formula is designed for something of the form  $(1 + u)^\alpha$ . Our bracket begins with 2, not 1, so we first factor the 2:

$$(2 - 3x)^{-2} = 2^{-2} \left(1 - \frac{3}{2}x\right)^{-2} = \frac{1}{4} \left(1 - \frac{3}{2}x\right)^{-2}.$$

Now use

$$(1 - z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \dots$$

with  $z = \frac{3}{2}x$ :

$$(2 - 3x)^{-2} = \frac{1}{4} + \frac{3}{4}x + \frac{27}{16}x^2 + \frac{27}{8}x^3 + \dots$$

The small quantity is  $\frac{3}{2}x$ , so

$$\left|\frac{3}{2}x\right| < 1 \quad \Rightarrow \quad \boxed{|x| < \frac{2}{3}}.$$

The factorization at the start is not cosmetic; it is what makes the generalized theorem applicable.

**HL Worked Example 18 - why  $\sqrt{37}$  must be rewritten**

Use a cubic binomial approximation to estimate  $\sqrt{37}$ . Explain why substituting  $x = 36$  into the expansion of  $\sqrt{1 + x}$  is not appropriate.

**HL Solution**

The expansion of  $(1 + x)^{1/2}$  is justified for  $|x| < 1$ . If we wrote  $\sqrt{37} = \sqrt{1 + 36}$ , then the small quantity would be 36, which is nowhere near the region where the series is valid. So that rewrite is mathematically inappropriate.

Instead, look for a nearby perfect square. Since 36 is close to 37,

$$\sqrt{37} = 6\sqrt{\frac{37}{36}} = 6\left(1 + \frac{1}{36}\right)^{1/2}.$$

Now the small quantity is  $1/36$ , which is genuinely small.

Using

$$(1+x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3,$$

with  $x = 1/36$  gives

$$\sqrt{37} \approx 6.082762667.$$

The success of the approximation comes from the rewrite, not merely from keeping more terms.

### HL Worked Example 19 - a fifth-root approximation

Expand  $(1+x)^{1/5}$  up to  $x^3$ . Hence approximate  $\sqrt[5]{1.01}$ , and explain why the same expansion is not justified for  $\sqrt[5]{2}$  by taking  $x = 1$ .

#### HL Solution

With  $\alpha = 1/5$ ,

$$(1+x)^{1/5} = 1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 + \dots$$

For  $\sqrt[5]{1.01}$ , take  $x = 0.01$ . This is well inside  $|x| < 1$ , so

$$\sqrt[5]{1.01} \approx \boxed{1.001992048}.$$

For  $\sqrt[5]{2}$ , writing  $2 = 1 + 1$  would require  $x = 1$ . The standard binomial series is not justified at the boundary  $|x| = 1$  in this IB setting, so we should not simply substitute 1. We would need a different rewrite around a more suitable reference value.

### HL Worked Example 20 - multiplying a finite factor by an infinite series

Expand

$$\frac{1+x^2}{\sqrt{1+x}}$$

up to and including  $x^3$ , and state the range of validity.

#### HL Solution

Rewrite the denominator as a negative power:

$$\frac{1+x^2}{\sqrt{1+x}} = (1+x^2)(1+x)^{-1/2}.$$

We only need terms up to  $x^3$ , so

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

Now multiply by  $1+x^2$ , keeping only powers at most 3:

$$\begin{aligned} (1+x^2)\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3\right) &= 1 - \frac{1}{2}x + \left(\frac{3}{8} + 1\right)x^2 \\ &\quad + \left(-\frac{5}{16} - \frac{1}{2}\right)x^3 + \dots \end{aligned}$$

Therefore

$$\frac{1+x^2}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{11}{8}x^2 - \frac{13}{16}x^3 + \dots$$

The infinite factor requires  $|x| < 1$ , and the finite factor adds no restriction. So the expansion is valid for

$$|x| < 1.$$

## 10 Past-paper patterns: what the question is hiding

IB-style binomial questions often look different on the surface, but many reduce to the same small set of decisions. The wording may mention a coefficient, an unknown parameter, a constant term or two neighbouring coefficients. Your first job is to identify which familiar structure is underneath.

### Worked Example 21 - parameter from one coefficient

In  $(x+k)^7$ , the coefficient of  $x^5$  is 84. Find the possible real values of  $k$ .

#### Solution

To obtain  $x^5$  from seven selections, we must choose  $x$  five times and  $k$  twice. Therefore the coefficient is

$$\binom{7}{2}k^2.$$

Now use the information supplied by the question:

$$21k^2 = 84.$$

Hence  $k^2 = 4$ , so

$$k = \pm 2.$$

The phrase “coefficient of  $x^5$ ” told us how many copies of  $k$  were needed before we used the number 84.

### Worked Example 22 - positive and negative powers in one bracket

In

$$\left(2x^2 - \frac{k}{x}\right)^8,$$

where  $k > 0$ , the coefficient of  $x^7$  is  $-14336$ . Find  $k$ .

#### Solution

The general exponent is

$$2(8-r) - r = 16 - 3r.$$

To create  $x^7$ ,

$$16 - 3r = 7 \Rightarrow r = 3.$$

Now we evaluate the coefficient of that term:

$$\binom{8}{3}2^{8-3}(-k)^3 = -1792k^3.$$

The question tells us that this equals  $-14336$ , so

$$-1792k^3 = -14336 \Rightarrow k^3 = 8.$$

## 10.1 Further past-paper patterns: four more structures

These four worked examples are **materially reconstructed IB-style adaptations**: the mathematical structures come from recurring AA past-paper/questionbank patterns already used in the CF Maths source documents, while the expressions, numerical values and wording are changed. The point is to learn the structure, not memorise an official question.

### Worked Example 23 - recover an integer exponent

In  $(2 + x^2)^{n+1}$ , where  $n \in \mathbb{Z}^+$ , the coefficient of  $x^4$  is 672. Find  $n$ .

To produce  $x^4$ , we must select  $x^2$  exactly twice. Therefore the required coefficient is

$$\binom{n+1}{2} 2^{(n+1)-2} = \binom{n+1}{2} 2^{n-1}.$$

So

$$\binom{n+1}{2} 2^{n-1} = 672.$$

Because  $n$  is a positive integer, test the discrete structure rather than treating this as an ordinary continuous equation. For  $n = 6$ ,

$$\binom{7}{2} 2^5 = 21 \cdot 32 = 672.$$

Hence  $\boxed{n = 6}$ . The key decision was that  $x^4$  comes from two selections of  $x^2$ .

### Worked Example 24 - a constant term hidden among positive and negative powers

Find the term independent of  $x$  in

$$\left(x^2 - \frac{3}{x}\right)^6.$$

If the second term is selected  $r$  times, the power of  $x$  is

$$2(6 - r) - r = 12 - 3r.$$

A constant term requires  $12 - 3r = 0$ , so  $r = 4$ . Only now evaluate the term:

$$\binom{6}{4} (x^2)^2 \left(-\frac{3}{x}\right)^4 = 15 \cdot 81 = \boxed{1215}.$$

Notice that the sign disappears because the required number of negative selections is even.

## 10.1 Further past-paper patterns - continued

### Worked Example 25 - compare neighbouring coefficients before expanding

In  $(2 + x)^n$ , where  $n \geq 3$  is an integer, the coefficient of  $x^3$  is twice the coefficient of  $x^2$ . Find  $n$ .

The two coefficients are

$$C_2 = \binom{n}{2} 2^{n-2}, \quad C_3 = \binom{n}{3} 2^{n-3}.$$

Instead of expanding the combinations separately, divide:

$$\frac{C_3}{C_2} = \frac{\binom{n}{3}}{\binom{n}{2}} \cdot \frac{1}{2} = \frac{n-2}{3} \cdot \frac{1}{2} = \frac{n-2}{6}.$$

The condition  $C_3 = 2C_2$  gives

$$\frac{n-2}{6} = 2 \Rightarrow n = \boxed{14}.$$

This ratio method is one of the quickest ways to deal with neighbouring coefficients.

### Worked Example 26 - a product of two finite binomial expansions

Find the coefficient of  $x^5$  in

$$(1 + x)^6(1 + 2x)^4$$

without writing either full expansion.

To make total degree 5, the two selected powers must add to 5. The valid pairs are

$$(1, 4), (2, 3), (3, 2), (4, 1), (5, 0).$$

Therefore the coefficient is

$$\binom{6}{1} \binom{4}{4} 2^4 + \binom{6}{2} \binom{4}{3} 2^3 + \binom{6}{3} \binom{4}{2} 2^2 + \binom{6}{4} \binom{4}{1} 2 + \binom{6}{5} \binom{4}{0}.$$

Evaluating only these contributions gives

$$96 + 480 + 480 + 120 + 6 = \boxed{1182}.$$

The exam habit here is: *list the power pairs first*. That prevents unnecessary expansion and missing contributions.

**Past-paper pattern check.** Across Examples 23–26, the surface wording changes, but the underlying decisions are only four: locate a power, set an exponent to zero, compare coefficient ratios, or list power pairs in a product.

## 10.2 Cross-topic synthesis: sequences meet the binomial theorem

### Worked Example 27 - normalised coefficients reveal a geometric sequence

Let  $c_r$  be the coefficient of  $x^r$  in

$$P(x) = (2 - ax)^8, \quad a > 0,$$

and define

$$u_{r+1} = \frac{c_r}{\binom{8}{r}}, \quad r = 0, 1, \dots, 8.$$

- Show that  $u_1, u_2, \dots, u_9$  form a geometric sequence. State its first term and common ratio in terms of  $a$ .
- Given that  $u_3 = 144$ , find  $a$ .
- Hence find the exact value of  $S_9 = u_1 + u_2 + \dots + u_9$ .
- Find the coefficient of  $x^4$  in  $P(x)$  without expanding the polynomial.
- Let  $w_n = |u_n|$  and extend this geometric rule for all  $n \geq 1$ . Find  $\sum_{n=1}^{\infty} w_n$  and the first  $n$  for which  $w_n < 50$ .

**Solution.** By the binomial theorem,

$$c_r = \binom{8}{r} 2^{8-r} (-a)^r.$$

Dividing by the binomial coefficient removes the combinatorial multiplicity:

$$u_{r+1} = 2^{8-r} (-a)^r = 256 \left(-\frac{a}{2}\right)^r.$$

Therefore  $(u_n)$  is geometric with

$$u_1 = 256, \quad q = -\frac{a}{2}.$$

Now  $u_3$  corresponds to  $r = 2$ , so

$$144 = u_3 = 256 \left(\frac{a^2}{4}\right) = 64a^2.$$

Since  $a > 0$ ,  $\boxed{a = \frac{3}{2}}$ , and hence  $q = -\frac{3}{4}$ .

For the first nine terms,

$$S_9 = 256 \frac{1 - (-3/4)^9}{1 + 3/4} = \boxed{\frac{40261}{256}}.$$

Also

$$u_5 = 256 \left(-\frac{3}{4}\right)^4 = 81,$$

so the coefficient of  $x^4$  is

$$c_4 = \binom{8}{4} u_5 = 70 \cdot 81 = \boxed{5670}.$$

For  $w_n = |u_n|$ ,

$$w_n = 256 \left(\frac{3}{4}\right)^{n-1},$$

so

$$\sum_{n=1}^{\infty} w_n = \frac{256}{1 - 3/4} = \boxed{1024}.$$

Since  $w_6 = 60.75$  but  $w_7 = 45.5625$ , the first term below 50 is  $\boxed{w_7}$ .

The useful idea is that dividing away the binomial-counting factor exposes a hidden geometric sequence; the  $u_n$  are not the original coefficients themselves.

## 10.3 HL MATERIAL - two further worked examples

### HL Worked Example 28 - a parameter inside a generalized binomial series

In  $(1 + ax)^{-3/2}$ , where  $a \neq 0$ , the coefficient of  $x^3$  is seven times the coefficient of  $x^2$ .

- Find  $a$ .
- State the interval of validity of the resulting infinite expansion.

For  $\alpha = -\frac{3}{2}$ , the coefficient of  $x^2$  is

$$\frac{\alpha(\alpha-1)}{2}a^2 = \frac{15}{8}a^2,$$

while the coefficient of  $x^3$  is

$$\frac{\alpha(\alpha-1)(\alpha-2)}{6}a^3 = -\frac{35}{16}a^3.$$

The condition is therefore

$$-\frac{35}{16}a^3 = 7\left(\frac{15}{8}a^2\right).$$

Since  $a \neq 0$ , divide by  $a^2$ :

$$-\frac{35}{16}a = \frac{105}{8} \Rightarrow \boxed{a = -6}.$$

The small quantity is now  $-6x$ , so validity requires

$$|-6x| < 1 \Rightarrow \boxed{|x| < \frac{1}{6}}.$$

This example is a good reminder that an unknown parameter can affect *both* the coefficients and the convergence interval.

### HL checkpoint

For generalized binomial questions, the coefficient comparison and the validity statement are not separate topics: once the parameter is found, it changes the small quantity  $u(x)$  and therefore changes the interval where the series represents the function.

## 10.3 HL MATERIAL - continued

### HL Worked Example 29 - product of two infinite binomial series and a GDC check

Let

$$F(x) = \left( \frac{1+2x}{1-x} \right)^{1/2} = (1+2x)^{1/2}(1-x)^{-1/2}.$$

- (a) Expand  $F(x)$  up to and including the term in  $x^3$ .
- (b) State the interval of validity.
- (c) Use  $x = 0.02$  to approximate  $\sqrt{1.04/0.98}$ , then compare with a GDC value.

We need only cubic terms:

$$\begin{aligned}(1+2x)^{1/2} &= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots, \\ (1-x)^{-1/2} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots.\end{aligned}$$

Multiply by matching powers only. The coefficient of  $x$  is

$$1 + \frac{1}{2} = \frac{3}{2}.$$

The coefficient of  $x^2$  is

$$-\frac{1}{2} + \frac{1}{2} + \frac{3}{8} = \frac{3}{8}.$$

The coefficient of  $x^3$  is

$$\frac{1}{2} - \frac{1}{4} + \frac{3}{8} + \frac{5}{16} = \frac{15}{16}.$$

Hence

$$F(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 + \frac{15}{16}x^3 + \cdots.$$

The two infinite factors require  $|2x| < 1$  and  $|x| < 1$ , so their intersection is

$$\boxed{|x| < \frac{1}{2}}.$$

With  $x = 0.02$ ,

$$F(0.02) \approx 1 + \frac{3}{2}(0.02) + \frac{3}{8}(0.02)^2 + \frac{15}{16}(0.02)^3 = \boxed{1.0301575}.$$

A GDC gives

$$\sqrt{\frac{1.04}{0.98}} \approx 1.0301575073,$$

so the absolute error is approximately  $7.28 \times 10^{-9}$ . The very small error is consistent with using a small value of  $x$  well inside the validity interval.

## 11 Final strategy sheet

### Finite powers - the core routine

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r.$$

1. Identify the two choices inside the bracket, including signs and powers.
2. Build the exponent of the variable you care about.
3. Solve for  $r$  **before** evaluating a coefficient.
4. Remember:  $r$  counts second-term selections; the term number is  $r + 1$ .
5. For a constant term, set the exponent equal to 0.
6. If  $r$  is not an integer in the allowed range, that power does not occur.
7. In a product, list only power pairs whose exponents add to the target.
8. Simplify algebraic structure before expanding.

### HL MATERIAL - infinite powers

1. Rewrite into  $A(1 + u)^\alpha$  and factor constants first.
2. Expand only as far as the requested degree.
3. State  $|u| < 1$  and intersect validity intervals for products.
4. For approximations, rewrite the target so that  $u$  is genuinely small.
5. If coefficients themselves form a sequence, use their ratio or a recurrence instead of recomputing each coefficient from scratch.
6. Never treat convergence as an afterthought; it is part of the mathematics.

**Locate the power first. Evaluate the coefficient second.**