

Binomial Theorem

Worked Solutions

Questions 1–32 • SL + clearly marked HL material

How these solutions are written. The required power is identified before arithmetic is carried out. Parameter questions are converted into coefficient equations, and HL infinite-series questions state the validity condition. The numbering matches the corrected 32-question worksheet.

Section A - Targeted terms and constant terms

1. Coefficient of x^5 in $(2 + 3x)^8$. The x^5 term uses five copies of $3x$:

$$\binom{8}{5} 2^3 (3x)^5.$$

Hence the coefficient is

$$\binom{8}{5} 2^3 3^5 = 56 \cdot 8 \cdot 243 = \boxed{108864}.$$

2. Complete term containing x^7 in $(x - 2)^9$. In $\binom{9}{r} x^{9-r} (-2)^r$, set $9 - r = 7$, so $r = 2$. Thus

$$\binom{9}{2} x^7 (-2)^2 = 36 \cdot 4x^7 = \boxed{144x^7}.$$

3. Constant term in $(x^2 + \frac{2}{x})^9$. The general power of x is

$$2(9 - r) - r = 18 - 3r.$$

For a constant term, $18 - 3r = 0$, giving $r = 6$. Therefore

$$\binom{9}{6} 2^6 = 84 \cdot 64 = \boxed{5376}.$$

4. Does $(x^3 + \frac{4}{x^2})^8$ have a constant term? The power is

$$3(8 - r) - 2r = 24 - 5r.$$

A constant would require $r = 24/5$, not an integer. Therefore there is no constant term.

5. Coefficient of x^{-2} in $(x + 1)^5 (\frac{1}{x} - 2x)^4$. If the first factor contributes x^i and the second factor uses r copies of $-2x$, then the total power is

$$i + (2r - 4).$$

Set this equal to -2 : $i + 2r = 2$. The only possibilities are $(i, r) = (2, 0)$ and $(0, 1)$. Hence

$$\binom{5}{2} + \binom{5}{0} \binom{4}{1} (-2) = 10 - 8 = \boxed{2}.$$

6. Fourth term in $(2x - a)^9$. The fourth term has $r = 3$:

$$\binom{9}{3} (2x)^6 (-a)^3 = -5376a^3 x^6.$$

Comparing with $-5376x^6$ gives $a^3 = 1$. Since $a > 0$, $a = 1$.

7. Coefficient of x^3 in $(2 + kx)^8$.

$$\binom{8}{3} 2^5 k^3 = 56 \cdot 32 k^3 = 1792k^3.$$

Thus $1792k^3 = 14336$, so $k^3 = 8$ and $k = 2$.

Section B - Recovering parameters and exponents

8. In $(1 + ax)^7$,

$$[x^4] = \binom{7}{4}a^4 = 35a^4, \quad [x^3] = \binom{7}{3}a^3 = 35a^3.$$

Given $[x^4] = 3[x^3]$,

$$35a^4 = 105a^3.$$

Since $a \neq 0$, $\boxed{a = 3}$.

9. $\binom{n}{2} = 66$, so

$$\frac{n(n-1)}{2} = 66 \Rightarrow n^2 - n - 132 = 0 = (n-12)(n+11).$$

Since $n \in \mathbb{Z}^+$, $\boxed{n = 12}$.

10. For $(1 + 2x)^n$,

$$[x^4] = 16\binom{n}{4}, \quad [x^3] = 8\binom{n}{3}.$$

The condition gives

$$16\binom{n}{4} = 6 \cdot 8\binom{n}{3}.$$

Using $\binom{n}{4}/\binom{n}{3} = (n-3)/4$,

$$16\frac{n-3}{4} = 48 \Rightarrow 4(n-3) = 48,$$

so $\boxed{n = 15}$.

11.

(a) The coefficient equations are

$$\binom{n}{2}a^2 = 135, \quad \binom{n}{3}a^3 = 540.$$

Divide the second by the first:

$$\frac{\binom{n}{3}}{\binom{n}{2}}a = 4.$$

Since $\binom{n}{3}/\binom{n}{2} = (n-2)/3$,

$$\frac{n-2}{3}a = 4 \Rightarrow \boxed{a(n-2) = 12}.$$

(b) Hence $a = 12/(n-2)$. Substituting into $\binom{n}{2}a^2 = 135$ gives the positive-integer solution $n = 6$. Then $a = 12/4 = 3$. Thus $\boxed{n = 6, a = 3}$.

12.

(a)

$$[x^2] = \binom{6}{2}2^4b^2 = 15 \cdot 16b^2 = 240b^2.$$

Given this is $240b$, $b^2 = b$. Since $b \neq 0$, $\boxed{b = 1}$.

(b)

$$[x^3] = \binom{6}{3}2^3 = 20 \cdot 8 = \boxed{160}.$$

13. In $(1 + px)^9$,

$$[x^4] = \binom{9}{4} p^4 = 126p^4, \quad [x^5] = \binom{9}{5} p^5 = 126p^5.$$

Equality and $p > 0$ give $\boxed{p = 1}$.

14. In $(2 + x)^n$,

$$[x^3] = \binom{n}{3} 2^{n-3}, \quad [x^2] = \binom{n}{2} 2^{n-2}.$$

The condition is

$$\binom{n}{3} 2^{n-3} = 2 \binom{n}{2} 2^{n-2}.$$

Divide by $\binom{n}{2} 2^{n-3}$:

$$\frac{n-2}{3} = 4,$$

so $\boxed{n = 14}$.

CF MATHS

Section C - Products of expansions and coefficient structure

15. Coefficient of x^5 in $(1+x)^6(1+2x)^4$. The powers must add to 5. The possible splits are $(1, 4), (2, 3), (3, 2), (4, 1), (5, 0)$. Therefore

$$\begin{aligned} [x^5] &= \binom{6}{1} \binom{4}{4} 2^4 + \binom{6}{2} \binom{4}{3} 2^3 + \binom{6}{3} \binom{4}{2} 2^2 \\ &\quad + \binom{6}{4} \binom{4}{1} 2 + \binom{6}{5} \\ &= 96 + 480 + 480 + 120 + 6 = \boxed{1182}. \end{aligned}$$

16. First simplify:

$$(1+x)^{10}(1-x)^{10} = (1-x^2)^{10}.$$

The x^8 term comes from $(-x^2)^4$, so the coefficient is

$$\binom{10}{4} (-1)^4 = \boxed{210}.$$

17. Let $P(x) = (1+2x)^7(1-x)^3$.

(a) For x^2 , use the splits $(0, 2), (1, 1), (2, 0)$:

$$1 \cdot 3 + 14(-3) + 84 = 3 - 42 + 84 = \boxed{45}.$$

(b) For x^3 , use $(0, 3), (1, 2), (2, 1), (3, 0)$:

$$1(-1) + 14(3) + 84(-3) + 280 = -1 + 42 - 252 + 280 = \boxed{69}.$$

(c) Full expansion is correct, but it calculates many coefficients that cannot contribute to the requested power. The power-split method keeps only the terms that can add to the target exponent.

Question 18 - IB-style SL synthesis

18. $F(x) = (1+ax)^6$, $a > 0$, and $[x^2]F = 240$.

(a) $\binom{6}{2}a^2 = 15a^2 = 240 \Rightarrow a^2 = 16$. Hence $\boxed{a = 4}$.

(b) $[x^3]F = \binom{6}{3}a^3 = 20 \cdot 64 = \boxed{1280}$.

(c) The coefficient of x^2 in $G = F(1+bx)$ is

$$240 + b[x]F = 240 + b(6a) = 240 + 24b.$$

Set equal to zero: $\boxed{b = -10}$.

(d)

$$[x^3]G = [x^3]F + b[x^2]F = 1280 - 10(240) = \boxed{-1120}.$$

(e) The choice $b = -10$ cancels the x^2 coefficient because that coefficient receives contributions from F_2 and bF_1 . The x^3 coefficient instead combines F_3 with bF_2 , so the same b need not cancel it.

Question 19 - Cross-topic SL synthesis: sequences meet binomial

19. $P_n(x) = (1 + 2x)^n$ and u_n is the sum of its coefficients.

(a) Setting $x = 1$ adds every coefficient, so

$$u_n = P_n(1) = (1 + 2)^n = \boxed{3^n}.$$

(b) Thus $u_0 = 1$ and $u_{n+1}/u_n = 3$. Therefore (u_n) is geometric with $\boxed{u_0 = 1, r = 3}$.

(c)

$$\sum_{n=0}^8 u_n = \sum_{n=0}^8 3^n = \frac{3^9 - 1}{3 - 1} = \boxed{9841}.$$

(d) The coefficient of x^2 is

$$\binom{n}{2} 2^2 = 4 \binom{n}{2} = 264.$$

Hence $\binom{n}{2} = 66$, giving $\boxed{n = 12}$.

(e) For $n = 12$, the sum of coefficients is $3^{12} = \boxed{531441}$. Also

$$\sum_{j=0}^{12} u_j = \frac{3^{13} - 1}{2} = \boxed{797161}.$$

(f) The alternating sum is obtained by substituting $x = -1$:

$$v_n = P_n(-1) = (1 - 2)^n = \boxed{(-1)^n}.$$

Thus (v_n) is geometric with first term 1 and common ratio -1 .

HL MATERIAL - Questions 20–26

20. Use $(1 + u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 + \dots$ with $u = 3x$:

$$\boxed{(1 + 3x)^{1/2} = 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \dots}.$$

Validity: $|3x| < 1 \Rightarrow \boxed{|x| < 1/3}$.

21. Rewrite

$$(2 - x)^{-2} = \frac{1}{4} \left(1 - \frac{x}{2}\right)^{-2}.$$

Since $(1 - u)^{-2} = 1 + 2u + 3u^2 + 4u^3 + \dots$,

$$\boxed{(2 - x)^{-2} = \frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \dots}.$$

Validity: $\boxed{|x| < 2}$.

22. Write

$$\left(\frac{4+x}{4-2x}\right)^{1/2} = (1 + x/4)^{1/2} (1 - x/2)^{-1/2}.$$

Expanding both factors to x^3 and multiplying gives

$$1 + \frac{3}{8}x + \frac{15}{128}x^2 + \frac{51}{1024}x^3 + \dots$$

The two conditions are $|x| < 4$ and $|x| < 2$, so a common interval is $|x| < 2$.

23. GDC. With $x = 0.04$,

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

Thus

$$1 + 0.02 - 0.0002 + 0.000004 = 1.019804.$$

The calculator value is $1.019803902718\dots$, so the absolute error is approximately

$$9.73 \times 10^{-8}.$$

24. $H(x) = (1 + cx)^{-1}(1 - x)^{1/2}$, $c > 0$.

(a) The linear terms are $-cx$ and $-\frac{1}{2}x$, so

$$-c - \frac{1}{2} = -\frac{9}{2} \Rightarrow \boxed{c = 4}.$$

(b) Up to x^2 ,

$$(1 + 4x)^{-1} = 1 - 4x + 16x^2 + \dots,$$

$$(1 - x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \dots.$$

Therefore

$$[x^2]H = 16 + (-4)(-1/2) - 1/8 = 18 - 1/8 = \boxed{\frac{143}{8}}.$$

(c) We need $|4x| < 1$ and $|x| < 1$, hence $\boxed{|x| < 1/4}$.

25. $F(x) = (1 + ax)^{-1/2}$ and $[x]F = -3$.

(a) The linear coefficient is $-a/2$, so $-a/2 = -3$ and $\boxed{a = 6}$.

(b) Using $u = 6x$,

$$\boxed{F(x) = 1 - 3x + \frac{27}{2}x^2 - \frac{135}{2}x^3 + \dots}.$$

(c) $|6x| < 1$, so $\boxed{|x| < 1/6}$.

(d) Since

$$(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \dots,$$

we obtain

$$[x^3]G = -\frac{135}{2} + 2\left(\frac{27}{2}\right) + 4(-3) + 8 = \boxed{-\frac{89}{2}}.$$

(e) Hence the cubic approximation is

$$G(x) \approx 1 - x + \frac{23}{2}x^2 - \frac{89}{2}x^3.$$

At $x = 0.01$,

$$\boxed{G(0.01) \approx 0.991106} \quad (\text{to 6 d.p.}).$$

The calculator value of $1/(\sqrt{1.06} \cdot 0.98)$ is $0.9911080228\dots$, giving absolute error

$$\boxed{2.52 \times 10^{-6}}.$$

(f) The conditions are $|6x| < 1$ and $|2x| < 1$. Both hold for $x = 0.01$. Because successive powers of 0.01 rapidly shrink, omitted terms beyond x^3 have a small numerical effect.

26. $F(x) = (1 - 2x)^{-1/2} = \sum c_r x^r$.

(a) Direct expansion gives

$$\boxed{c_0 = 1, \quad c_1 = 1, \quad c_2 = \frac{3}{2}, \quad c_3 = \frac{5}{2}}.$$

(b)

$$c_r = \binom{-1/2}{r} (-2)^r.$$

Therefore

$$\frac{c_{r+1}}{c_r} = \frac{-\frac{1}{2} - r}{r+1} (-2) = \boxed{\frac{2r+1}{r+1}}.$$

Hence

$$\boxed{c_{r+1} = c_r \frac{2r+1}{r+1}}, \quad c_0 = 1.$$

(c)

$$c_4 = \frac{5}{2} \cdot \frac{7}{4} = \boxed{\frac{35}{8}}, \quad c_5 = \frac{35}{8} \cdot \frac{9}{5} = \boxed{\frac{63}{8}}.$$

(d)

$$\frac{t_{r+1}}{t_r} = 0.01 \frac{2r+1}{r+1}.$$

Since $(2r+1)/(r+1) < 2$, this ratio is less than 0.02. Thus each successive term is less than 2% of the preceding one, so the numerical series decreases extremely rapidly.

(e)

$$\begin{aligned} S_5 &= 1 + 0.01 + \frac{3}{2}(0.01)^2 + \frac{5}{2}(0.01)^3 + \frac{35}{8}(0.01)^4 + \frac{63}{8}(0.01)^5 \\ &= 1.0101525445375 \dots = \boxed{1.01015254} \text{ (8 d.p.)}. \end{aligned}$$

The calculator value of $1/\sqrt{0.98}$ is 1.01015254455221.... Using the unrounded S_5 , the absolute error is approximately

$$\boxed{1.47 \times 10^{-11}}.$$

(f) The infinite expansion requires $|-2x| < 1$, i.e. $\boxed{|x| < 1/2}$. Since $0.01 < 1/2$, the chosen value is valid.

Section F - Mixed challenge and exam finishers

27. In $(x + k)^8$, the x^6 term is

$$\binom{8}{2} x^6 k^2 = 28k^2 x^6.$$

Thus $28k^2 = 448$, so $k^2 = 16$ and $k = \pm 4$.

28. In $\left(x^2 - \frac{3}{x}\right)^9$, the general power is $18 - 3r$. Set to zero: $r = 6$. Therefore the constant term is

$$\binom{9}{6} (-3)^6 = 84 \cdot 729 = 61236.$$

29. Since

$$A(x) = (1 + x)^n (1 - x)^n = (1 - x^2)^n,$$

(a) The x^8 coefficient is $\binom{n}{4}$. Hence $\binom{n}{4} = 70$, so $n = 8$.

(b) The x^6 term uses $(-x^2)^3$:

$$\binom{8}{3} (-1)^3 = -56.$$

30. For $(1 + qx)^{-2}$,

$$1 - 2qx + 3q^2x^2 - 4q^3x^3 + \dots$$

(a) $-2q = -8$, so $q = 4$.

(b) The x^3 coefficient is $-4q^3 = -4(64) = -256$.

(c) Validity: $|qx| < 1 \Rightarrow |x| < 1/4$.

31. Up to x^4 ,

$$(1 + 2x)^{-1/2} = 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \frac{35}{8}x^4 + \dots,$$

while $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$. Therefore the x^4 coefficient is the sum of the first five coefficients of the first factor:

$$1 - 1 + \frac{3}{2} - \frac{5}{2} + \frac{35}{8} = \frac{27}{8}.$$

32. The expansion $(1 + x)^{1/2}$ is an infinite binomial series valid only for $|x| < 1$. Substituting $x = 8$ lies far outside that interval, so the truncated expression does not represent $\sqrt{9}$ there.

A suitable rewrite is

$$3 = \sqrt{9} = \sqrt{8 \left(1 + \frac{1}{8}\right)} = 2\sqrt{2} \left(1 + \frac{1}{8}\right)^{1/2}.$$

Now the small parameter is $x = 1/8$, which satisfies $|x| < 1$, so a binomial approximation is legitimate.

Final habit. Identify the required power first. In HL infinite expansions, state the validity interval before trusting a numerical approximation.