

Binomial Theorem Homework Worksheet

The Binomial Theorem

Questions only • Increasing difficulty • Full working expected

Instructions. Attempt the questions in order. Show algebraic working clearly, identify the required power before evaluating a coefficient, and justify “show that” statements. Questions marked **GDC** are intended for calculator use. The worksheet contains both SL and clearly marked HL material, including two cross-topic synthesis questions connecting sequences with the binomial theorem.

Source note. The questions are original CF Maths teaching material written in an IB-style format. Past-paper structures are paraphrased or materially reconstructed rather than reproduced verbatim.

Questions run continuously from **1 to 32**. Questions 1–19 are SL-accessible; Questions 20–26 form the dedicated HL sequence; Questions 27–32 are mixed finishers with HL items clearly marked.

Section A - Targeted terms and constant terms

Choose the term before calculating it. In $(a + b)^n$, the general term may be written as $\binom{n}{r} a^{n-r} b^r$.

1. Find the coefficient of x^5 in $(2 + 3x)^8$. [3]

2. Find the complete term containing x^7 in $(x - 2)^9$. [3]

3. Find the term independent of x in

$$\left(x^2 + \frac{2}{x}\right)^9.$$

[5]

4. Determine whether

$$\left(x^3 + \frac{4}{x^2}\right)^8$$

contains a term independent of x . Justify your answer without fully expanding.

[4]

5. Find the coefficient of x^{-2} in

$$(x + 1)^5 \left(\frac{1}{x} - 2x\right)^4.$$

[7]

6. The fourth term in the expansion of $(2x - a)^9$ is $-5376x^6$. Find the positive value of a . [5]

7. In $(2 + kx)^8$, the coefficient of x^3 is 14336. Find the real value of k . [5]

Checkpoint: Questions 6 and 7 are not “expansion” questions. They are equations disguised as coefficient statements.

Section B - Recovering parameters and exponents

These questions reverse the usual direction. Use coefficient information to determine an unknown parameter, an exponent, or both.

8. In $(1 + ax)^7$, where $a \neq 0$, the coefficient of x^4 is three times the coefficient of x^3 . Find a . [5]
9. The coefficient of x^2 in $(1 + x)^n$ is 66, where $n \in \mathbb{Z}^+$. Find n . [4]
10. In $(1 + 2x)^n$, where $n > 3$, the coefficient of x^4 is 6 times the coefficient of x^3 . Find n . [5]
11. In $(1 + ax)^n$, where $a > 0$ and $n \in \mathbb{Z}^+$, the coefficient of x^2 is 135 and the coefficient of x^3 is 540.
- (a) Show that $a(n - 2) = 12$. [2]
- (b) Hence find n and a . [5]
12. In $(2 + bx)^6$, the coefficient of x^2 is $240b$.
- (a) Find the non-zero value of b . [4]
- (b) For this value of b , find the coefficient of x^3 . [2]
13. The coefficient of x^4 in $(1 + px)^9$ is equal to the coefficient of x^5 in the same expansion. Given $p > 0$, find p . [5]
14. The coefficient of x^3 in $(2 + x)^n$ is twice the coefficient of x^2 . Given $n \geq 3$ is an integer, find n . [5]

Section C - Products of expansions and coefficient structure

When two expansions are multiplied, the coefficient of x^m is the sum of all products whose powers add to m .

15. Without fully expanding, find the coefficient of x^5 in

$$(1+x)^6(1+2x)^4.$$

[6]

16. Find the coefficient of x^8 in

$$(1+x)^{10}(1-x)^{10}$$

by simplifying the product first.

[4]

17. Let

$$P(x) = (1+2x)^7(1-x)^3.$$

(a) Find the coefficient of x^2 in $P(x)$.

[5]

(b) Find the coefficient of x^3 in $P(x)$.

[6]

(c) Explain why evaluating both full expansions first is mathematically correct but inefficient.

[2]

CF MATHS

IB-style SL synthesis - Question 18

This is one connected problem. Each part should feed naturally into the next; do not restart from the beginning each time.

18. Let $F(x) = (1 + ax)^6$, where $a > 0$. The coefficient of x^2 in $F(x)$ is 240.

- (a) Find a . [3]
- (b) Hence find the coefficient of x^3 in $F(x)$. [2]
- (c) Let $G(x) = F(x)(1 + bx)$. Given that the coefficient of x^2 in $G(x)$ is zero, find b . [4]
- (d) Hence find the coefficient of x^3 in $G(x)$. [3]
- (e) Explain, using your answers, why multiplying by a linear factor can eliminate one coefficient without eliminating the next coefficient. [2]

Cross-topic SL synthesis - Question 19: sequences meet the binomial theorem

This connected problem links the sum of coefficients of a binomial expansion with a geometric sequence and geometric series. Reuse each result instead of starting again.

19. For each integer $n \geq 0$, let $P_n(x) = (1 + 2x)^n$, and let u_n be the sum of all coefficients in the expansion of $P_n(x)$.

(a) Explain why $u_n = P_n(1)$, and hence show that $u_n = 3^n$. [3]

(b) Hence show that $u_0, u_1, u_2, \dots$ is a geometric sequence, stating its first term and common ratio. [2]

(c) Find the exact value of

$$\sum_{n=0}^8 u_n.$$

[3]

(d) The coefficient of x^2 in $P_n(x)$ is 264. Show that $n = 12$. [4]

(e) For this value of n , find the sum of all coefficients in $P_n(x)$ and then find

$$\sum_{j=0}^{12} u_j.$$

[4]

(f) Let v_n be the alternating sum of the coefficients of $P_n(x)$; that is, coefficients of even powers are added and coefficients of odd powers are subtracted. By evaluating $P_n(-1)$, find v_n and explain why (v_n) is also geometric. [3]

HL MATERIAL - Section D - Generalized binomial theorem

For $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots,$$

valid for $|x| < 1$ when the series is infinite. State the interval of validity whenever appropriate.

20. HL MATERIAL. Find the first four terms of $(1+3x)^{1/2}$ and state the interval of validity. [6]

21. HL MATERIAL. Expand $(2-x)^{-2}$ up to and including the term in x^3 . State the interval of validity. [7]

22. HL MATERIAL. Expand

$$\left(\frac{4+x}{4-2x}\right)^{1/2}$$

up to and including the term in x^3 . State a valid interval for x . [9]

23. HL MATERIAL GDC. Using a binomial expansion up to x^3 , obtain an approximation to $\sqrt{1.04}$. Give your answer to six decimal places. Then use your GDC to find the calculator value and state the absolute error of your approximation. [7]

Why validity matters

A truncated series can look algebraically convincing even when the substitution lies outside the interval in which the infinite binomial series represents the function. A good HL solution states this explicitly rather than treating the condition as decorative.

HL MATERIAL - Section E - Coefficients, products and approximation

These questions combine generalized expansions with coefficient reasoning. Keep only as many terms as the requested power requires.

24. HL MATERIAL. Let

$$H(x) = (1 + cx)^{-1}(1 - x)^{1/2}, \quad c > 0.$$

The coefficient of x in the expansion of $H(x)$ is $-\frac{9}{2}$.

- (a) Find c . [3]
- (b) Hence find the coefficient of x^2 in $H(x)$. [5]
- (c) State the interval of x for which both infinite factors are represented by their binomial series. [2]

IB-style HL synthesis - Question 25

A long connected task using parameter recovery, generalized expansion, product coefficients, validity and approximation.

25. HL MATERIAL. Let

$$F(x) = (1 + ax)^{-1/2}, \quad a > 0.$$

The coefficient of x in the expansion of $F(x)$ is -3 .

- (a) Show that $a = 6$. [2]
- (b) Hence expand $F(x)$ up to and including the term in x^3 . [5]
- (c) State the interval of validity of this expansion. [2]
- (d) Define

$$G(x) = F(x)(1 - 2x)^{-1}.$$

Find the coefficient of x^3 in the expansion of $G(x)$. [5]

- (e) **GDC** Use your cubic expansions to approximate

$$\frac{1}{\sqrt{1.06} (0.98)}$$

by taking $x = 0.01$. Give your approximation to six decimal places, then use your GDC to calculate the corresponding exact numerical value and the absolute error. [5]

- (f) Explain why $x = 0.01$ is valid for both infinite series and comment on why terms beyond x^3 should have a relatively small effect on this approximation. [3]

HL cross-topic synthesis - Question 26: a coefficient sequence from an infinite binomial expansion

This problem treats the coefficients themselves as a recursively defined sequence, then uses a partial sum as a numerical approximation.

26. HL MATERIAL. Consider

$$F(x) = (1 - 2x)^{-1/2} = \sum_{r=0}^{\infty} c_r x^r,$$

where c_r denotes the coefficient of x^r .

(a) Use the generalized binomial theorem to find c_0, c_1, c_2, c_3 . [5]

(b) Starting from the general coefficient, show that

$$\frac{c_{r+1}}{c_r} = \frac{2r+1}{r+1},$$

and hence write a recurrence relation for c_{r+1} in terms of c_r . [5]

(c) Use the recurrence, rather than re-expanding $F(x)$, to find c_4 and c_5 . [3]

(d) Define $t_r = c_r(0.01)^r$. Show that

$$\frac{t_{r+1}}{t_r} = 0.01 \frac{2r+1}{r+1} < 0.02$$

for every $r \geq 0$, and explain what this tells you about the successive terms used in the approximation below. [4]

(e) **GDC** Calculate

$$S_5 = \sum_{r=0}^5 c_r (0.01)^r$$

and use it as an approximation to $1/\sqrt{0.98}$. Give S_5 to eight decimal places, then use your GDC to find the absolute error. [5]

(f) State the interval of validity of the original infinite binomial expansion and verify that $x = 0.01$ lies in it. [2]

Final strategy

For a difficult binomial question, ask in this order: **(1)** what power is required? **(2)** which term or pair of terms can create it? **(3)** is there an unknown parameter that turns the coefficient into an equation? **(4)** if the exponent is not a non-negative integer, what is the validity condition? **(5)** can a previous part be reused instead of starting again?

Section F - Mixed challenge and exam finishers

These final problems are short but intentionally non-routine. They are suitable as extension, homework selection or timed finishers.

27. In $(x + k)^8$, where $k \in \mathbb{R}$, the coefficient of x^6 is 448. Find the possible values of k . [5]

28. Find the term independent of x in

$$\left(x^2 - \frac{3}{x}\right)^9.$$

[6]

29. Let $A(x) = (1 + x)^n(1 - x)^n$, where $n \in \mathbb{Z}^+$. The coefficient of x^8 in $A(x)$ is 70.

(a) Find n . [4]

(b) Hence find the coefficient of x^6 in $A(x)$. [3]

30. **HL MATERIAL.** The first three non-zero terms in the expansion of $(1 + qx)^{-2}$ are

$$1 - 8x + 48x^2 + \dots$$

(a) Find q . [2]

(b) Find the coefficient of x^3 . [3]

(c) State the interval of validity. [2]

31. **HL MATERIAL.** Find the coefficient of x^4 in

$$(1 + 2x)^{-1/2}(1 - x)^{-1}$$

without multiplying more terms than necessary. [7]

32. A student claims that because $(1 + x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$, substituting $x = 8$ gives a useful approximation to 3. Explain precisely why this reasoning is invalid, then give a suitable rewrite of $3 = \sqrt{9}$ that would allow a small-parameter binomial approximation. [5]

End of worksheet. Questions 1–19 are SL-accessible; Questions 20–26 are the dedicated HL sequence; Questions 27–32 are mixed finishers with HL items clearly marked.