

CF MATHS

IB Mathematics AA Lecture Notes

Deductive Proof

Equality proofs • even and odd integers • squares and cubes • divisibility • linked “show that” arguments

How these notes are written. A proof should read like a mathematical story. We do not jump from a formula to the answer. We say what we are trying to show, choose a useful representation, explain why each step is allowed, and only then write the conclusion. When proving an equality, we normally transform one side into the other rather than manipulating both sides at once.

Focused chapter scope. What deduction means; the difference between checking and proving; how to represent even, odd and consecutive integers; how to prove an equality from LHS to RHS; squares and cubes; divisibility; factorisation; domain restrictions; planning a proof backwards but writing it forwards; and IB-style linked questions in which one “show that” result is used in the next part.

What is deductive proof?

A deductive proof begins with facts that are already known and uses logically valid steps to reach the statement we want to establish. The important word is *general*. Trying a few numerical examples can suggest that a statement is true, but it does not prove that the statement works for every permitted value.

Suppose somebody claims that the sum of two odd integers is even. Testing $3 + 5 = 8$ and $7 + 11 = 18$ is useful for spotting the pattern, but there are infinitely many odd integers. A proof must handle all of them at once.

The way we do that is to translate words into algebra.

Representations you should know

For integers $k, m \in \mathbb{Z}$:

$$\text{even integer} = 2k,$$

$$\text{odd integer} = 2k + 1,$$

$$\text{consecutive integers} = k, k + 1,$$

$$\text{consecutive odd integers} = 2k + 1, 2k + 3,$$

$$\text{consecutive even integers} = 2k, 2k + 2.$$

To prove that an expression is even, we try to rewrite it as $2 \times (\text{an integer})$. To prove that it is odd, we try to rewrite it as $2 \times (\text{an integer}) + 1$.

Key insight. The letter k is not a particular number. It stands for an arbitrary integer. That is what makes the argument general.

Worked Example 1 - the sum of two even integers

Prove that the sum of any two even integers is even.

Solution. Let the two even integers be $2a$ and $2b$, where $a, b \in \mathbb{Z}$. We now add them:

$$2a + 2b = 2(a + b).$$

Because $a + b$ is also an integer, the expression has the form $2 \times (\text{an integer})$. Therefore the sum is even. $\square$

Notice the final sentence. The algebra alone is not quite the whole proof; we explicitly connect the final form to the definition of an even integer.

Worked Example 2 - the sum of two odd integers

Prove that the sum of any two odd integers is even.

Solution. Write the odd integers as $2a + 1$ and $2b + 1$. Then

$$(2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1).$$

The quantity $a + b + 1$ is an integer, so the sum is twice an integer. Hence the sum is even. $\square$

Worked Example 3 - an even integer multiplied by an odd integer

Prove that the product of an even integer and an odd integer is even.

Solution. Let the even integer be $2a$ and the odd integer be $2b + 1$. Their product is

$$(2a)(2b + 1) = 2(a(2b + 1)).$$

Since $a(2b + 1)$ is an integer, the product is of the form $2q$ for some integer q . Therefore it is even. $\square$

Common trap. Writing “let the odd integer be $2k$ ” destroys the proof immediately. The representation

must match the property you are trying to use.

Equality proofs: how to go from LHS to RHS

This is the proof style the IB guide describes as transforming a left-hand side into a right-hand side, or vice versa. The safest presentation is usually:

$$\text{LHS} = \dots = \dots = \text{RHS}.$$

We choose one side - usually the messier side - and simplify it until it becomes the other side.

IB exam habit. Do *not* begin by writing “LHS = RHS” and then perform the same operation on both sides. That assumes the statement you were asked to prove. It is excellent for solving an equation, but it is poor proof-writing because it can become circular.

A four-step routine

1. Decide which side is easier to transform.
2. Look at the target. Does it suggest factoring, a common denominator, expanding a square, or using an identity?
3. Transform only the chosen side, justifying any non-obvious step.
4. When the target appears, write “= RHS, as required.”

Trick. It is completely acceptable to work backwards on scrap paper to discover the route. The final written proof should normally run forwards from the given side to the required side.

Worked Example 4 - rational equality

For $x \neq \pm 1$, prove that

$$\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{4x}{x^2-1}.$$

Solution. The left-hand side is more complicated, so we start there. The two fractions have different denominators, so the natural first move is to create a common denominator:

$$\begin{aligned} \text{LHS} &= \frac{(x+1)^2 - (x-1)^2}{(x-1)(x+1)} \\ &= \frac{(x^2 + 2x + 1) - (x^2 - 2x + 1)}{x^2 - 1} \\ &= \frac{4x}{x^2 - 1} \\ &= \text{RHS}. \end{aligned}$$

Hence the identity is proved for $x \neq \pm 1$. □

Why this works. The restrictions $x \neq \pm 1$ matter because the original fractions are undefined there.

Worked Example 5 - a difference of squares hiding inside an identity

Prove that

$$(a+b)^2 - (a-b)^2 = 4ab.$$

Solution. We could expand both squares, but there is a faster structure. The left side is a difference

of two squares:

$$\begin{aligned}(a+b)^2 - (a-b)^2 &= ((a+b) - (a-b))((a+b) + (a-b)) \\ &= (2b)(2a) \\ &= 4ab.\end{aligned}$$

So the left-hand side has become the right-hand side. $\square$

Worked Example 6 - a trigonometric equality

For values of x for which both sides are defined, prove that

$$\frac{1 - \cos^2 x}{\sin x} = \sin x.$$

Solution. The expression $1 - \cos^2 x$ should immediately suggest the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1.$$

Therefore

$$\frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x,$$

as required, provided $\sin x \neq 0$. $\square$

Key insight. In an identity proof, do not ask “what can I do?” Ask “what structure would make the target appear?” That question is much more useful.

Squares, cubes and parity

Powers often look different from basic even/odd questions, but the method is the same: substitute the general form of the integer and simplify until the parity becomes visible.

Worked Example 7 - the square of an odd integer is odd

Prove that the square of any odd integer is odd.

Solution. Let the odd integer be $2k + 1$. Then

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, the square has the form $2q + 1$. Therefore it is odd. $\square$

Worked Example 8 - the square of an even integer is divisible by 4

Let n be even. Prove that n^2 is divisible by 4.

Solution. Because n is even, write $n = 2k$. Then

$$n^2 = (2k)^2 = 4k^2.$$

As $k^2 \in \mathbb{Z}$, this is 4 multiplied by an integer. Hence $4 \mid n^2$. $\square$

Worked Example 9 - consecutive odd squares

Prove that the difference between the squares of two consecutive odd integers is divisible by 8.

Solution. Consecutive odd integers can be written as $2k + 1$ and $2k + 3$. We want the difference of their squares:

$$\begin{aligned}(2k + 3)^2 - (2k + 1)^2 &= ((2k + 3) - (2k + 1))((2k + 3) + (2k + 1)) \\ &= 2(4k + 4) \\ &= 8(k + 1).\end{aligned}$$

Because $k + 1$ is an integer, the difference is divisible by 8. □

Trick. Here factorisation is much cleaner than expanding both squares. Whenever you see $A^2 - B^2$, consider $(A - B)(A + B)$ first.

Divisibility proofs

The notation $d \mid N$ means that $N = dq$ for some integer q . So a divisibility proof is really a factoring problem: we try to expose the required divisor.

Worked Example 10 - proving $n^3 - n$ is divisible by 6

Prove that $n^3 - n$ is divisible by 6 for every integer n .

Solution. Start by factoring:

$$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1).$$

These are three consecutive integers. Among any three consecutive integers, one is divisible by 3. Also, among any two consecutive integers, one is even, so the product is divisible by 2. Therefore the product is divisible by both 2 and 3, and hence by 6. Thus

$$6 \mid (n^3 - n).$$

□

Why this works. The key step is not “ $n^3 - n$ looks divisible by 6”. The key step is exposing three consecutive integer factors.

Worked Example 11 - an odd square is one more than a multiple of 8

Let n be odd. Prove that $n^2 - 1$ is divisible by 8.

Solution. Write $n = 2k + 1$. Then

$$\begin{aligned}n^2 - 1 &= (2k + 1)^2 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k + 1).\end{aligned}$$

The integers k and $k + 1$ are consecutive, so one of them is even. Hence $k(k + 1) = 2m$ for some integer m . Therefore

$$n^2 - 1 = 4(2m) = 8m,$$

which proves that $8 \mid (n^2 - 1)$. □

Linked proofs: “show that ... hence ...”

Many IB questions do not ask for an isolated proof. Part (a) asks you to show a useful identity or equation; part (b) then expects you to reuse it. The result in part (a) is not decoration - it is usually the intended shortcut.

Worked Example 12 - one equality creates the next one

Let $x \neq 0$ and define

$$s = x + \frac{1}{x}.$$

(a) Show that

$$x^2 + \frac{1}{x^2} = s^2 - 2.$$

(b) Show that

$$x^3 + \frac{1}{x^3} = s^3 - 3s.$$

(c) Hence, if $s = 3$, find $x^2 + x^{-2}$ and $x^3 + x^{-3}$.

Solution.

(a) Start with the definition of s and square it:

$$\begin{aligned} s^2 &= \left(x + \frac{1}{x}\right)^2 \\ &= x^2 + 2 + \frac{1}{x^2}. \end{aligned}$$

Now rearrange:

$$x^2 + \frac{1}{x^2} = s^2 - 2.$$

(b) Cube the same expression:

$$\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.$$

The middle two terms are $3s$, so

$$s^3 = x^3 + \frac{1}{x^3} + 3s.$$

Hence

$$x^3 + \frac{1}{x^3} = s^3 - 3s.$$

(c) Now the earlier results do all the work:

$$x^2 + \frac{1}{x^2} = 3^2 - 2 = 7,$$

$$x^3 + \frac{1}{x^3} = 3^3 - 3(3) = 18.$$

Answer: 7 and 18.

IB exam habit. When a question says *hence*, look back. The previous result should normally save time and reduce algebra.

IB-style deductive proof patterns: keep the algebra central

In an examination, a deductive proof may appear inside a larger question, but the proof itself is often quite elementary. The most useful practice is therefore not to attach proof to an unrelated topic, but to become comfortable with algebraic statements that naturally lead from one result to the next.

The next two examples deliberately stay close to the core ideas of this chapter: factorisation, polynomial structure, integer arguments and the word *hence*. They are original CF Maths questions written in the style of linked IB “show that ... hence” tasks.

Worked Example 13 - a polynomial statement that unlocks the next parts

Let

$$p(x) = x^3 + ax^2 + bx - 6,$$

where $a, b \in \mathbb{R}$. Suppose that $p(1) = 0$ and $p(2) = 0$.

(a) Show that

$$a + b = 5$$

and

$$2a + b = -1.$$

(b) Hence find a and b .

(c) Hence factorise $p(x)$ completely and solve $p(x) = 0$.

Solution.

(a) The information $p(1) = 0$ means that substituting $x = 1$ into the polynomial must give zero:

$$1 + a + b - 6 = 0.$$

Therefore

$$a + b = 5.$$

Now use $p(2) = 0$:

$$8 + 4a + 2b - 6 = 0.$$

Dividing by 2 gives

$$2a + b = -1.$$

Both required relations have now been obtained directly from the given root information.

(b) Subtract the first equation from the second:

$$(2a + b) - (a + b) = -1 - 5,$$

so

$$a = -6.$$

Then $a + b = 5$ gives

$$b = 11.$$

(c) We now know

$$p(x) = x^3 - 6x^2 + 11x - 6.$$

Because $p(1) = 0$ and $p(2) = 0$, the factors $(x - 1)$ and $(x - 2)$ must occur. The remaining factor is $(x - 3)$, giving

$$p(x) = (x - 1)(x - 2)(x - 3).$$

Hence

$$p(x) = 0 \implies x = 1, 2, 3.$$

Key insight. The important proof habit is that each part supplies something useful for the next one. Do not restart the problem from scratch after a “hence”.

Worked Example 14 - four consecutive integers and a hidden square

Let $n \in \mathbb{Z}$.

(a) Show that

$$n(n + 1)(n + 2)(n + 3) + 1 = (n^2 + 3n + 1)^2.$$

(b) Hence prove that the product of any four consecutive integers, increased by 1, is a perfect square.

(c) Hence find the integer values of n for which

$$n(n+1)(n+2)(n+3) + 1 = 361.$$

Solution.

(a) Expanding four brackets at once would work, but it hides the structure. Instead pair the first and fourth factors, and the middle two factors:

$$n(n+3) = n^2 + 3n,$$

$$(n+1)(n+2) = n^2 + 3n + 2.$$

Let

$$t = n^2 + 3n.$$

Then

$$\begin{aligned} n(n+1)(n+2)(n+3) + 1 &= t(t+2) + 1 \\ &= t^2 + 2t + 1 \\ &= (t+1)^2 \\ &= (n^2 + 3n + 1)^2. \end{aligned}$$

This proves the identity.

(b) Since n is an integer, $n^2 + 3n + 1$ is also an integer. Part (a) therefore writes the required number as the square of an integer. Hence it is a perfect square.

(c) Using the result from part (a),

$$(n^2 + 3n + 1)^2 = 361 = 19^2.$$

Therefore

$$n^2 + 3n + 1 = 19$$

or

$$n^2 + 3n + 1 = -19.$$

The first gives

$$n^2 + 3n - 18 = 0 = (n-3)(n+6),$$

so $n = 3$ or $n = -6$. The second equation has no integer solutions. Thus

$$\boxed{n = 3 \text{ or } n = -6}.$$

Final proof checklist

Before you finish a proof, ask:

1. Did I define my integers correctly? For example, $\text{odd} = 2k + 1$ with $k \in \mathbb{Z}$.
2. Did I prove a general statement rather than test examples?
3. If proving an equality, did I transform one side into the other rather than assume they were equal?
4. If dividing or cancelling, did I check that the quantity is non-zero?
5. If proving divisibility, did I reach $d \times (\text{an integer})$?
6. Did I explain why the final form proves the claim?
7. In a linked question, did I use the result from the earlier part when the word *hence* appeared?