

CF MATHS

IB Mathematics AA

Deductive Proof - Worked Solutions

How these solutions are written. The reasoning is stated before the algebra. Each proof identifies the representation or identity being used, then explains why the final form completes the proof. In equality questions, one side is transformed into the other rather than assuming the equality at the start.

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Section A - even, odd and consecutive integers

Solution 1 - even + even

Let the integers be $2a$ and $2b$. Then

$$2a + 2b = 2(a + b).$$

Since $a + b \in \mathbb{Z}$, the result is twice an integer, so it is even. □

Solution 2 - odd + even

Let the odd integer be $2a + 1$ and the even integer be $2b$. Then

$$(2a + 1) + 2b = 2(a + b) + 1.$$

This is of the form $2q + 1$, so it is odd. □

Solution 3 - odd times odd

Let the odd integers be $2a + 1$ and $2b + 1$. Their product is

$$(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1.$$

The bracket is an integer, so the product is odd. □

Solution 4 - square of an odd integer

Let $n = 2k + 1$. Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence n^2 is odd. □

Solution 5 - square of an even integer

Let $n = 2k$. Then

$$n^2 = 4k^2.$$

Since k^2 is an integer, $4 \mid n^2$. □

Solution 6 - consecutive sum

Let the consecutive integers be n and $n + 1$. Then

$$n + (n + 1) = 2n + 1,$$

which is odd. □

Solution 7 - consecutive product

For consecutive integers n and $n + 1$, one must be even. Therefore their product $n(n + 1)$ contains a factor 2, so it is even. □

Solution 8 - difference of consecutive squares

$$(n + 1)^2 - n^2 = 2n + 1.$$

This has the form $2q + 1$, so the difference is odd. □

Section B - squares, cubes and divisibility**Solution 9 - odd square minus 1**

Let $n = 2k + 1$. Then

$$n^2 - 1 = 4k^2 + 4k = 4k(k + 1).$$

Because k and $k + 1$ are consecutive, $k(k + 1)$ is even. Write $k(k + 1) = 2m$. Then

$$n^2 - 1 = 8m,$$

so it is divisible by 8. □

Solution 10 - $n^3 - n$

Factor:

$$n^3 - n = n(n - 1)(n + 1).$$

These are three consecutive integers. One is divisible by 3, and at least one is even. Hence the product is divisible by $2 \cdot 3 = 6$. □

Solution 11 - consecutive odd squares

Let the odd integers be $2k + 1$ and $2k + 3$. Then

$$(2k + 3)^2 - (2k + 1)^2 = 2(4k + 4) = 8(k + 1).$$

Hence the difference is divisible by 8. □

Solution 12 - consecutive odd cubes

Let the integers be $2k + 1$ and $2k + 3$. Each odd cube is odd, but we need the stronger statement “divisible by 4”. Expand only as far as needed:

$$\begin{aligned} (2k + 1)^3 + (2k + 3)^3 &= (8k^3 + 12k^2 + 6k + 1) \\ &\quad + (8k^3 + 36k^2 + 54k + 27) \\ &= 16k^3 + 48k^2 + 60k + 28 \\ &= 4(4k^3 + 12k^2 + 15k + 7). \end{aligned}$$

The bracket is an integer, so the sum is divisible by 4. □

Solution 13 - three consecutive integers

The product $n(n + 1)(n + 2)$ contains three consecutive integers. One is divisible by 3, and at least one is even. Therefore the product is divisible by 6. □

Solution 14 - cube of an even integer

Let $n = 2k$. Then

$$n^3 = 8k^3,$$

so $8 \mid n^3$. □

Solution 15 - odd n : $24 \mid (n^3 - n)$

Factor:

$$n^3 - n = n(n - 1)(n + 1).$$

The three factors are consecutive, so one is divisible by 3. Since n is odd, both $n - 1$ and $n + 1$ are even. They are consecutive even integers, so one is divisible by 4 and the other by 2. Thus the product contains a factor $3 \cdot 4 \cdot 2 = 24$. Hence $24 \mid (n^3 - n)$. □

Section C - proving equalities**Solution 16 - rational identity**

The restrictions are $x \neq \pm 2$. Start from the left:

$$\begin{aligned} \frac{x+2}{x-2} - \frac{x-2}{x+2} &= \frac{(x+2)^2 - (x-2)^2}{(x-2)(x+2)} \\ &= \frac{8x}{x^2 - 4}. \end{aligned}$$

Thus LHS = RHS. □

Solution 17 - difference of squares

$$\begin{aligned} (a+b)^2 - (a-b)^2 &= ((a+b) - (a-b))((a+b) + (a-b)) \\ &= (2b)(2a) = 4ab. \end{aligned}$$

□

Solution 18 - cube identity

Expand the left side:

$$\begin{aligned} (a+b)^3 + (a-b)^3 &= (a^3 + 3a^2b + 3ab^2 + b^3) \\ &\quad + (a^3 - 3a^2b + 3ab^2 - b^3) \\ &= 2a^3 + 6ab^2 \\ &= 2a(a^2 + 3b^2). \end{aligned}$$

Hence LHS = RHS. □

Solution 19 - trigonometric identity

We require $\sin x \neq 0$. Using $1 - \cos^2 x = \sin^2 x$,

$$\frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x.$$

□

Solution 20 - rational identity

The restrictions are $x \neq \pm 1$.

$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{(x+1) - (x-1)}{(x-1)(x+1)} = \frac{2}{x^2-1}.$$

□

Solution 21 - binomial coefficient identity

Using factorial definitions,

$$\frac{n}{r} \binom{n-1}{r-1} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!(n-r)!}.$$

Now $n(n-1)! = n!$ and $r(r-1)! = r!$, so

$$\frac{n}{r} \binom{n-1}{r-1} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Therefore the identity holds.

□

Section D - linked proofs**Solution 22 - powers of $x + x^{-1}$**

(a)

$$s^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2},$$

so

$$x^2 + x^{-2} = s^2 - 2.$$

(b)

$$s^3 = \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right).$$

Therefore

$$x^3 + x^{-3} = s^3 - 3s.$$

(c) With $s = 4$,

$$x^2 + x^{-2} = 16 - 2 = 14,$$

$$x^3 + x^{-3} = 64 - 12 = 52.$$

Answer: 14 and 52.

Solution 23 - consequences of an odd square

(a) From Solution 9, $n^2 - 1$ is divisible by 8. Therefore $n^2 - 1 = 8q$ for some integer q , so

$$n^2 = 8q + 1.$$

(b) Then

$$n^2 + 7 = (8q + 1) + 7 = 8(q + 1),$$

so it is divisible by 8.

(c) Factor:

$$n^4 - 1 = (n^2 - 1)(n^2 + 1).$$

Part (a) shows that $n^2 - 1$ is divisible by 8. Since n is odd, n^2 is odd, so $n^2 + 1$ is even. The product is therefore divisible by $8 \cdot 2 = 16$. $\square$

Solution 24 - quotient plus remainder form

(a) Divide $x^2 + 1$ by $x - 1$:

$$x^2 + 1 = (x - 1)(x + 1) + 2.$$

Therefore

$$f(x) = x + 1 + \frac{2}{x - 1}.$$

(b) Solve

$$x + 1 + \frac{2}{x - 1} = 5.$$

Then

$$x - 4 + \frac{2}{x - 1} = 0.$$

Multiplying by $x - 1$ (allowed because $x \neq 1$):

$$(x - 4)(x - 1) + 2 = 0,$$

$$x^2 - 5x + 6 = 0,$$

$$(x - 2)(x - 3) = 0.$$

Thus $x = 2$ or $x = 3$. Both satisfy $x \neq 1$.

Section E - difficult deductive proof challenges

Solution 25 - root information determines a polynomial

(a) Since $p(1) = 0$,

$$1 + a + b - 6 = 0,$$

so

$$a + b = 5.$$

Since $p(2) = 0$,

$$8 + 4a + 2b - 6 = 0.$$

Dividing by 2 gives

$$2a + b = -1.$$

(b) Subtract the first relation from the second:

$$a = -6.$$

Then $a + b = 5$ gives

$$b = 11.$$

(c) Thus

$$p(x) = x^3 - 6x^2 + 11x - 6.$$

We already know that $x = 1$ and $x = 2$ are roots, so $(x - 1)$ and $(x - 2)$ are factors. The remaining factor is $(x - 3)$:

$$p(x) = (x - 1)(x - 2)(x - 3).$$

(d) Therefore

$$p(x) = 0 \implies x = 1, 2, 3.$$

Solution 26 - completing a square inside a polynomial

(a)

$$(2x + 1)^2 + 3 = 4x^2 + 4x + 1 + 3 = 4(x^2 + x + 1) = 4A.$$

Hence

$$4A = (2x + 1)^2 + 3.$$

(b) For every real x ,

$$(2x + 1)^2 \geq 0.$$

Therefore

$$(2x + 1)^2 + 3 \geq 3 > 0.$$

Since this equals $4A$ and $4 > 0$, we must have

$$A > 0.$$

(c) Part (b) proves that $x^2 + x + 1$ is strictly positive for every real x . It can therefore never equal zero. Hence the equation has no real solutions.

(d) Factor the difference of cubes:

$$x^3 - 1 = (x - 1)(x^2 + x + 1).$$

For $x \neq 1$,

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1 = A.$$

Part (b) shows $A > 0$, so

$$\frac{x^3 - 1}{x - 1} > 0$$

for every real $x \neq 1$.

Solution 27 - repeated consequences of oddness

(a) Let $n = 2k + 1$. Then

$$\begin{aligned} n^2 - 1 &= (2k + 1)^2 - 1 \\ &= 4k(k + 1). \end{aligned}$$

Because k and $k + 1$ are consecutive, one is even, so $k(k + 1) = 2m$ for some integer m . Hence

$$n^2 - 1 = 8m,$$

so it is divisible by 8.

(b) From part (a), write $n^2 - 1 = 8m$. Then

$$(n^2 - 1)^2 = 64m^2,$$

which is divisible by 64.

(c) Since

$$n^4 - 2n^2 + 1 = (n^2 - 1)^2,$$

part (b) immediately proves that $n^4 - 2n^2 + 1$ is divisible by 64.

(d) From part (a), $n^2 = 8m + 1$. Therefore

$$\begin{aligned} n^4 + 6n^2 + 1 &= (8m + 1)^2 + 6(8m + 1) + 1 \\ &= 64m^2 + 64m + 8 \\ &= 8(8m^2 + 8m + 1). \end{aligned}$$

The bracket is an integer, so the expression is divisible by 8.

Solution 28 - a hidden perfect square

(a) Pair the factors strategically:

$$\begin{aligned} n(n + 3) &= n^2 + 3n, \\ (n + 1)(n + 2) &= n^2 + 3n + 2. \end{aligned}$$

Let $t = n^2 + 3n$. Then

$$\begin{aligned} n(n + 1)(n + 2)(n + 3) + 1 &= t(t + 2) + 1 \\ &= t^2 + 2t + 1 \\ &= (t + 1)^2 \\ &= (n^2 + 3n + 1)^2. \end{aligned}$$

(b) Since $n \in \mathbb{Z}$, the number $n^2 + 3n + 1$ is an integer. Part (a) therefore expresses the given quantity as the square of an integer, so it is a perfect square.

(c) Part (a) gives

$$(n^2 + 3n + 1)^2 = 361 = 19^2.$$

Thus

$$n^2 + 3n + 1 = 19$$

or

$$n^2 + 3n + 1 = -19.$$

The first equation gives

$$n^2 + 3n - 18 = 0 = (n - 3)(n + 6),$$

so $n = 3$ or $n = -6$. The second equation,

$$n^2 + 3n + 20 = 0,$$

has no integer solutions. Therefore

$$\boxed{n = 3 \text{ or } n = -6}.$$

Solution 29 - consecutive odd integers

Let $a = 2k + 1$ and $b = 2k + 3$.

(a)

$$\begin{aligned} b^2 - a^2 &= (b - a)(b + a) \\ &= 2(4k + 4) \\ &= 8(k + 1). \end{aligned}$$

Hence the difference is divisible by 8.

(b)

$$\begin{aligned} a^3 + b^3 &= (2k + 1)^3 + (2k + 3)^3 \\ &= 16k^3 + 48k^2 + 60k + 28 \\ &= 4(4k^3 + 12k^2 + 15k + 7). \end{aligned}$$

The bracket is an integer, so the sum is divisible by 4.

(c) Since a and b are consecutive odd integers, $b = a + 2$. Then

$$a^2 + (a + 2)^2 = 130.$$

So

$$2a^2 + 4a + 4 = 130,$$

$$a^2 + 2a - 63 = 0,$$

$$(a + 9)(a - 7) = 0.$$

Thus $a = 7$ or $a = -9$. Hence the ordered pairs are

$$(a, b) = (7, 9) \quad \text{or} \quad (-9, -7).$$

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