

CF MATHS

IB Mathematics AA

Deductive Proof Worksheet

Core proof skills • equality • parity • powers • divisibility • linked “show that” questions

How to use this worksheet. Write complete proofs, not only algebra. Define variables, state why a final expression has the required form, and include domain restrictions when cancelling or dividing. In equality questions, transform one side into the other. Questions 22-29 contain linked proof and IB-style tasks in which later parts depend on earlier results.

Questions are original CF Maths material. The final section contains harder linked deductive-proof challenges focused on polynomials, parity, divisibility and perfect-square arguments.

CF MATHS

Section A - even, odd and consecutive integers

1. Prove that the sum of two even integers is even. [3]
2. Prove that the sum of an odd integer and an even integer is odd. [3]
3. Prove that the product of two odd integers is odd. [4]
4. Prove that the square of an odd integer is odd. [4]
5. Prove that the square of an even integer is divisible by 4. [3]
6. Prove that the sum of two consecutive integers is odd. [3]
7. Prove that the product of two consecutive integers is even. [3]
8. Prove that the difference between the squares of two consecutive integers is odd. [4]

Section B - squares, cubes and divisibility

9. Let n be odd. Prove that $n^2 - 1$ is divisible by 8. [5]
10. Prove that $n^3 - n$ is divisible by 6 for every integer n . [5]
11. Prove that the difference between the squares of two consecutive odd integers is divisible by 8. [5]
12. Prove that the sum of the cubes of two consecutive odd integers is divisible by 4. [6]
13. Prove that $n(n+1)(n+2)$ is divisible by 6 for every integer n . [5]
14. If n is even, prove that n^3 is divisible by 8. [3]
15. If n is odd, prove that $n^3 - n$ is divisible by 24. [7]

Section C - proving equalities

For each question, state any necessary restrictions.

16. Prove that

$$\frac{x+2}{x-2} - \frac{x-2}{x+2} = \frac{8x}{x^2-4}.$$
 [5]
17. Prove that

$$(a+b)^2 - (a-b)^2 = 4ab.$$
 [4]
18. Prove that

$$(a+b)^3 + (a-b)^3 = 2a(a^2 + 3b^2).$$
 [5]
19. Prove that

$$\frac{1 - \cos^2 x}{\sin x} = \sin x.$$
 [4]
20. Prove that

$$\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}.$$
 [4]

21. For integers $1 \leq r \leq n$, prove directly from factorial definitions that

$$\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1}.$$

[5]

Section D - linked proofs

22. Let $x \neq 0$ and $s = x + x^{-1}$.

(a) Show that $x^2 + x^{-2} = s^2 - 2$.

[3]

(b) Show that $x^3 + x^{-3} = s^3 - 3s$.

[4]

(c) Hence, if $s = 4$, find $x^2 + x^{-2}$ and $x^3 + x^{-3}$.

[2]

23. Let n be an odd integer.

(a) Show that $n^2 = 8q + 1$ for some integer q .

[5]

(b) Hence show that $n^2 + 7$ is divisible by 8.

[2]

(c) Hence show that $n^4 - 1$ is divisible by 16.

[4]

24. For $x \neq 1$, define

$$f(x) = \frac{x^2 + 1}{x - 1}.$$

(a) Show that

$$f(x) = x + 1 + \frac{2}{x - 1}.$$

[4]

(b) Hence solve $f(x) = 5$.

[4]

Section E - difficult deductive proof challenges

These final questions are original CF Maths problems. They stay deliberately close to the content of this chapter: polynomial identities, parity, divisibility, perfect squares and linked “show that ... hence” reasoning.

25. Let

$$p(x) = x^3 + ax^2 + bx - 6,$$

where $a, b \in \mathbb{R}$. It is known that $p(1) = 0$ and $p(2) = 0$.

(a) Show that $a + b = 5$ and $2a + b = -1$.

[4]

(b) Hence find a and b .

[3]

(c) Hence factorise $p(x)$ completely.

[3]

(d) Hence solve $p(x) = 0$.

[1]

26. For real x , let

$$A = x^2 + x + 1.$$

(a) Show that

$$4A = (2x + 1)^2 + 3.$$

[3]

(b) Hence prove that $A > 0$ for every real x .

[3]

- (c) Hence prove that the equation

$$x^2 + x + 1 = 0$$

has no real solutions, without using the quadratic formula.

[2]

- (d) For $x \neq 1$, show that

$$\frac{x^3 - 1}{x - 1} = A,$$

and hence determine the sign of $\frac{x^3 - 1}{x - 1}$.

[4]

27. Let n be an odd integer.

- (a) Show that $n^2 - 1$ is divisible by 8.

[5]

- (b) Hence show that $(n^2 - 1)^2$ is divisible by 64.

[2]

- (c) Hence prove that

$$n^4 - 2n^2 + 1$$

is divisible by 64.

[2]

- (d) Show also that

$$n^4 + 6n^2 + 1$$

is divisible by 8.

[4]

28. Let $n \in \mathbb{Z}$.

- (a) Show that

$$n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2.$$

[6]

- (b) Hence prove that the product of any four consecutive integers, increased by 1, is a perfect square.

[2]

- (c) Hence find all integer values of n for which

$$n(n+1)(n+2)(n+3) + 1 = 361.$$

[5]

29. Let a and b be consecutive odd integers, with $b > a$.

- (a) Show that $b^2 - a^2$ is divisible by 8.

[5]

- (b) Show that $b^3 + a^3$ is divisible by 4.

[5]

- (c) If $a^2 + b^2 = 130$, find a and b .

[4]