

IB Mathematics AA Lecture Notes

Sequences, Arithmetic and Geometric Sequences

What this chapter is trying to do

We are not going to memorise a long list of formulas. We will first learn to **see the pattern**, then decide whether the change is additive or multiplicative, and only then choose the formula. Throughout the chapter the IB notation u_1 is used for the first term.

Focused scope. General sequence notation and simple recursion • the link between u_n and S_n • arithmetic and geometric consecutive-term methods • arithmetic and geometric sequences and series • finite and infinite sums • finding an index n • simultaneous sequence conditions • GDC tables and thresholds • IB-style mixed AP/GP problems.

Independent CF Maths teaching material. Not an official publication of the International Baccalaureate Organization.

Historic IB questions are used only as style references; wording and numerical data are reconstructed.

1 Start with the pattern, not the formula

1.1 What is a sequence?

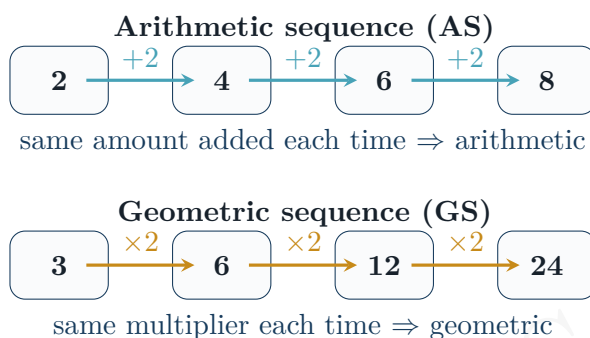
A sequence is an ordered list of numbers:

$$u_1, u_2, u_3, \dots$$

The subscript tells us the position. So u_5 means “the fifth term”. A **series** is what we get when we add terms:

$$S_n = u_1 + u_2 + \dots + u_n.$$

Before doing any algebra, ask one question: **what is happening from one term to the next?**



Key insight. Arithmetic means *add the same number*. Geometric means *multiply by the same number*. If neither happens, do not force the sequence into one of these two families.

Worked Example 1 • reading a general sequence

NO GDC

The sequence is defined by

$$u_n = n^2 - 3n + 5, \quad n = 1, 2, 3, \dots$$

Find u_1, u_2, u_5 , and then find the positive integer n for which $u_n = 27$.

Detailed solution

We are given the term directly, so we simply substitute the required index:

$$u_1 = 3, \quad u_2 = 3, \quad u_5 = 15.$$

For $u_n = 27$,

$$n^2 - 3n + 5 = 27 \iff n^2 - 3n - 22 = 0.$$

Its roots are $\frac{3 \pm \sqrt{97}}{2}$, neither of which is a positive integer. Hence no term is exactly 27.

Answer: $u_1 = 3, u_2 = 3, u_5 = 15$; **no positive integer n gives $u_n = 27$.**

1.2 Explicit rules and recursive rules

An **explicit rule** gives u_n directly from n . For example,

$$u_n = 4n - 1.$$

A **recursive rule** tells us how to get the next term from the previous one. For example,

$$u_1 = 5, \quad u_{n+1} = u_n + 3.$$

Now let us read this recurrence as a story: start at 5, then add 3, then add 3 again, and keep going.

$$5 \longrightarrow 8 \longrightarrow 11 \longrightarrow 14 \longrightarrow \dots$$

Worked Example 2 • making recursion feel simple

NO GDC

A sequence is defined by

$$u_1 = 7, \quad u_{n+1} = 2u_n - 1.$$

Find the first four terms. Is the sequence arithmetic or geometric?

Detailed solution

We do not need a complicated recurrence method. We just follow the instruction one step at a time.

$$u_2 = 2(7) - 1 = 13,$$

$$u_3 = 2(13) - 1 = 25,$$

$$u_4 = 2(25) - 1 = 49.$$

So the sequence begins

$$7, 13, 25, 49, \dots$$

The differences are 6, 12, 24, so they are not constant. The ratios are also not constant. Therefore the sequence is neither arithmetic nor geometric.

Answer: 7, 13, 25, 49; **neither arithmetic nor geometric.**

1.3 Moving between terms and partial sums

The notation S_n means the sum of the first n terms:

$$S_n = u_1 + u_2 + \cdots + u_n.$$

This gives one of the most useful identities in the whole chapter. Write two consecutive partial sums underneath one another:

$$\begin{aligned} S_n &= u_1 + u_2 + \cdots + u_{n-1} + u_n, \\ S_{n-1} &= u_1 + u_2 + \cdots + u_{n-1}. \end{aligned}$$

Everything before the last term cancels, so

$$\boxed{u_n = S_n - S_{n-1} \quad (n \geq 2)}.$$

For the first term we simply use

$$\boxed{u_1 = S_1}.$$

There is a second useful subtraction. If we want a block of terms from u_p to u_q , then

$$\boxed{u_p + u_{p+1} + \cdots + u_q = S_q - S_{p-1}}.$$

Think of S_q as containing everything up to q ; subtracting S_{p-1} removes everything before the block we actually want.

Worked Example 3 • recovering the sequence from S_n

NO GDC

A sequence has partial sums

$$S_n = \frac{n(3n+1)}{2}.$$

- Find u_1 .
- Use $u_n = S_n - S_{n-1}$ to find a formula for u_n .
- Hence show that the sequence is arithmetic and state d .
- Find $u_5 + u_6 + \cdots + u_{12}$ without adding the eight terms separately.

Detailed solution

(a) The first partial sum contains only the first term, so

$$u_1 = S_1 = \frac{1(3+1)}{2} = 2.$$

(b) Now we use the subtraction formula. For $n \geq 2$,

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= \frac{n(3n+1)}{2} - \frac{(n-1)(3n-2)}{2} \\ &= \frac{3n^2 + n - (3n^2 - 5n + 2)}{2} \\ &= 3n - 1. \end{aligned}$$

The same rule also gives $u_1 = 2$, so it works for every $n \geq 1$.

(c) We now compare consecutive terms:

$$u_{n+1} - u_n = [3(n+1) - 1] - (3n - 1) = 3.$$

The difference is constant, so the sequence is arithmetic with $d = 3$.

(d) Here is the important shortcut. The sum from term 5 to term 12 is the first 12 terms with the first 4 removed:

$$u_5 + \cdots + u_{12} = S_{12} - S_4.$$

Now

$$S_{12} = \frac{12(37)}{2} = 222, \quad S_4 = \frac{4(13)}{2} = 26.$$

Therefore

$$u_5 + \cdots + u_{12} = 222 - 26 = 196.$$

Answer: $u_1 = 2$, $u_n = 3n - 1$, $d = 3$, **required sum** = 196.

Quick practice - partial sums

1. If $S_n = 2^n - 1$, find u_n and the least n for which $u_n > 1000$.

2. If $S_n = n^2 + 4n$, find u_{10} and $u_6 + u_7 + \cdots + u_{10}$.

Answers. 1. $u_n = 2^{n-1}$ and $n = 11$. 2. $u_n = 2n + 3$, so $u_{10} = 23$ and the required sum is $S_{10} - S_5 = 95$.

Common trap. Do not see the word “sequence” and immediately write an AP or GP formula. First identify the structure actually given.

2 Arithmetic sequences: the same amount is added

2.1 The arithmetic sequence formulas

Suppose the first term is u_1 and the common difference is d . Then

$$u_n = u_1 + (n - 1)d.$$

For the sum of the first n terms,

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d) \quad \text{or} \quad S_n = \frac{n}{2}(u_1 + u_n).$$

How to choose the formula

If the question asks for one term, start with u_n . If it asks for a total, start with S_n . If two pieces of information refer to the same AP, expect two equations in u_1 and d .

2.2 Consecutive terms in an arithmetic sequence

Use the **same notation throughout**. Let

$$a, b, c$$

be three consecutive terms of an arithmetic sequence. The common difference is the same between each pair:

$$d = b - a, \quad d = c - b.$$

Therefore the complete proof is

$$d = b - a, d = c - b \implies b - a = c - b \implies 2b = a + c \implies b = \frac{a + c}{2}.$$

So the middle term is the arithmetic mean of the two outer terms. Equivalently, for indexed terms,

$$2u_n = u_{n-1} + u_{n+1}.$$

If the middle term itself is known, the same relationship can be written symmetrically as

$$a = b - d, \quad b, \quad c = b + d.$$

Why this matters in IB questions

When three expressions are said to be consecutive AP terms, do *not* introduce several unrelated variables. Label them a, b, c in order and use

$$2b = a + c.$$

This converts the sequence condition into one algebraic equation immediately.

Worked Example 4 • consecutive AP terms

NO GDC

Three consecutive terms of an *increasing* arithmetic sequence have sum 45 and product 3315. Find the three terms and the common difference.

Detailed solution

Because the information is symmetric around the middle term, we write the terms as

$$x - d, \quad x, \quad x + d.$$

The sum is especially simple:

$$(x - d) + x + (x + d) = 3x = 45,$$

so

$$x = 15.$$

Now use the product:

$$(15 - d)(15)(15 + d) = 3315.$$

The outer factors are a difference of squares:

$$15(225 - d^2) = 3315.$$

Divide by 15:

$$225 - d^2 = 221 \Rightarrow d^2 = 4.$$

Since the sequence is stated to be increasing, $d > 0$, so $d = 2$. The terms are therefore

$$13, \quad 15, \quad 17.$$

Answer: Terms 13, 15, 17 and $d = 2$.

Quick practice - consecutive AP terms

The consecutive terms u_8, u_9, u_{10} have sum 72, and $u_{10} - u_8 = 10$. Find u_1 , d and u_{20} .

Answer. The middle term is $u_9 = 24$; also $2d = 10$, so $d = 5$. Hence $u_1 = 24 - 8(5) = -16$ and $u_{20} = 79$.

Worked Example 5 • find a term, then reverse to find n

NO GDC

An arithmetic sequence has $u_1 = 11$ and $d = 4$.

- (a) Find u_8 .
- (b) Find n if $u_n = 95$.
- (c) Find S_{12} .

Detailed solution

(a) We are moving from term 1 to term 8, so there are 7 jumps of size 4:

$$u_8 = 11 + 7(4) = 39.$$

(b) Now the term value is known but the position is not:

$$95 = 11 + (n - 1)4.$$

So

$$84 = 4(n - 1) \Rightarrow n - 1 = 21 \Rightarrow n = 22.$$

(c) For a sum we switch to the series formula:

$$S_{12} = \frac{12}{2}(2(11) + 11(4)) = 6(66) = 396.$$

Answer: $u_8 = 39$, $n = 22$, $S_{12} = 396$.

Worked Example 6 • two known terms give a simultaneous system

NO GDC

An arithmetic sequence satisfies

$$u_5 = 18, \quad u_{13} = 50.$$

Find u_1 , d and u_{20} .

Detailed solution

This is a very common IB structure. We have two unknowns, u_1 and d , so we need two equations. From $u_5 = 18$:

$$u_1 + 4d = 18.$$

From $u_{13} = 50$:

$$u_1 + 12d = 50.$$

Subtract the first equation from the second:

$$8d = 32 \Rightarrow d = 4.$$

Then

$$u_1 + 4(4) = 18 \Rightarrow u_1 = 2.$$

Finally,

$$u_{20} = 2 + 19(4) = 78.$$

Answer: $u_1 = 2$, $d = 4$, $u_{20} = 78$.

Exam trick. When two non-consecutive terms are given, subtracting the two equations usually eliminates u_1 immediately.

Worked Example 7 • IB-style: a term and a sum condition

NO GDC

An arithmetic sequence satisfies

$$u_8 = S_8 = 8.$$

Find u_1 and d .

Structure adapted from IB reference 21M.1.SL.TZ1.3; wording is reconstructed.

Detailed solution

Let us use the condition $u_8 = 8$ first:

$$u_1 + 7d = 8. \quad (1)$$

Now use $S_8 = 8$. Because u_8 is already known, the shortest sum formula is

$$S_8 = \frac{8}{2}(u_1 + u_8).$$

Therefore

$$8 = 4(u_1 + 8).$$

Divide by 4:

$$2 = u_1 + 8 \Rightarrow u_1 = -6.$$

Substitute into (1):

$$-6 + 7d = 8 \Rightarrow 7d = 14 \Rightarrow d = 2.$$

Answer: $u_1 = -6$, $d = 2$.

Worked Example 8 • find n from a sum

NO GDC

An arithmetic sequence has $u_1 = 5$ and $d = 3$. The sum of the first n terms is 590. Find n .

Detailed solution

Because the information is about a total, we start with S_n :

$$590 = \frac{n}{2}(2(5) + (n-1)3).$$

Simplify the bracket first:

$$590 = \frac{n}{2}(3n + 7).$$

Multiply by 2:

$$1180 = 3n^2 + 7n.$$

So

$$3n^2 + 7n - 1180 = 0.$$

Factorising gives

$$(3n + 59)(n - 20) = 0.$$

A sequence index must be positive, so

$$n = 20.$$

Answer: $n = 20$.

3 Arithmetic sequences with GDC and modelling

When a GDC is genuinely useful

A calculator is useful when the index cannot be isolated neatly, when a maximum/minimum of S_n must be found, or when two models need to be compared. It should not replace simple algebra that can be done exactly by hand.

Casio fx-CG50 - table method: MENU $\rightarrow$ TABLE. Enter the expression using X in place of n . Set **Start** = 1, choose a suitable **End**, and **Pitch** = 1. Scan the integer rows.

TI-Nspire CX/CX II - table method: open **Graphs**, enter the expression, then press **Ctrl+T**. Use **Menu** $\rightarrow$ **Table** $\rightarrow$ **Edit Table Settings** to set the table start and step. Use step 1 for sequence indices.

Worked Example 9 • maximum partial sum

GDC

An arithmetic sequence has

$$u_1 = 60, \quad d = -2.5.$$

(a) Find the position of the zero term.

(b) Find the maximum value of S_n .

Structure adapted from IB reference 21M.2.SL.TZ2.3.

Detailed solution

(a)

$$u_n = 60 + (n - 1)(-2.5).$$

Set this equal to zero:

$$60 - 2.5(n - 1) = 0 \Rightarrow n = 25.$$

So $u_{25} = 0$. The terms before this are positive; after this they are negative.

(b) The partial sum is

$$S_n = \frac{n}{2}(120 - 2.5(n - 1)).$$

A table of integer values shows that the greatest sum occurs at $n = 24$ or $n = 25$; adding the zero term does not change the total.

$$S_{24} = \frac{24}{2}(60 + 2.5) = 750,$$

and $S_{25} = 750$ as well.

The important interpretation is not just the calculator result. The sum grows while positive terms are being added, stays unchanged when the zero term is added, and then falls once negative terms appear.

Answer: Maximum partial sum = 750.

Worked Example 10 • modelling a changing quantity

NO GDC

A theatre has 28 seats in the first row. Each new row has 4 more seats than the previous row.

- (a) Find the number of seats in row 18.
- (b) Find the total number of seats in the first 18 rows.
- (c) Explain what $d = 4$ means in context.

Detailed solution

The rows form an arithmetic sequence because the change is a constant addition of 4.

$$u_1 = 28, \quad d = 4.$$

(a)

$$u_{18} = 28 + 17(4) = 96.$$

(b)

$$S_{18} = \frac{18}{2}(28 + 96) = 9(124) = 1116.$$

(c) The common difference is not just a number in a formula. It means *every row contains exactly four more seats than the row before it*.

Answer: $u_{18} = 96$ seats; total = 1116 seats.

IB-style bridge: when arithmetic sums form a geometric sequence

Thinking Example 1 • partial sums become consecutive GP terms

The first and fourth terms of an arithmetic sequence are a and b , respectively. The sum of the first n terms is S_n .

- Find the common difference d in terms of a and b .
- Find S_5 , S_6 and S_9 in terms of a and b .
- Given that S_5, S_6, S_9 are consecutive terms of a geometric sequence, show that

$$\boxed{2a^2 = 5b^2}.$$

Detailed solution

- (a) Since the fourth term is $a + 3d = b$,

$$\boxed{d = \frac{b - a}{3}}.$$

- (b) Use $S_n = \frac{n}{2}[2a + (n - 1)d]$ and substitute the same expression for d each time:

$$S_5 = \frac{5}{2} \left(2a + 4 \frac{b - a}{3} \right) = \boxed{\frac{5}{3}(a + 2b)},$$

$$S_6 = 3 \left(2a + 5 \frac{b - a}{3} \right) = \boxed{a + 5b},$$

$$S_9 = \frac{9}{2} \left(2a + 8 \frac{b - a}{3} \right) = \boxed{3(4b - a)}.$$

- (c) For three consecutive GP terms, the square of the middle term equals the product of the outer two:

$$S_6^2 = S_5 S_9.$$

Hence

$$(a + 5b)^2 = \frac{5}{3}(a + 2b) 3(4b - a) = 5(a + 2b)(4b - a).$$

Expand both sides:

$$a^2 + 10ab + 25b^2 = -5a^2 + 10ab + 40b^2.$$

The mixed terms cancel, leaving

$$6a^2 = 15b^2 \implies \boxed{2a^2 = 5b^2}.$$

The hidden move

The arithmetic sequence gives formulas for the *sums*; the geometric condition is then imposed on those sums. Always finish the AP algebra first, then use $\boxed{(\text{middle})^2 = (\text{first})(\text{third})}$.

IB-style bridge: a second layer of structure

Thinking Example 2 • finding the AP from its geometric partial sums

An arithmetic sequence has first term $u_1 = 4$ and non-zero common difference d . Its partial sums are S_n .

- Express S_2, S_4, S_8 in terms of d .
- Given that S_2, S_4, S_8 are consecutive terms of a geometric sequence, find d .
- Hence find the common ratio of this geometric sequence.
- Show that, for the value of d found, $S_n = 4n^2$. Hence predict S_{16} without reusing the arithmetic-series formula.

Detailed solution

For an AP with $u_1 = 4$,

$$S_n = \frac{n}{2}(8 + (n-1)d).$$

Therefore

$$S_2 = 8 + d, \quad S_4 = 16 + 6d, \quad S_8 = 32 + 28d.$$

Because these are consecutive GP terms,

$$S_4^2 = S_2 S_8.$$

So

$$(16 + 6d)^2 = (8 + d)(32 + 28d).$$

Expanding and simplifying gives

$$256 + 192d + 36d^2 = 256 + 256d + 28d^2,$$

$$8d^2 - 64d = 0 \implies 8d(d - 8) = 0.$$

Since $d \neq 0$,

$$\boxed{d = 8}.$$

Then

$$S_2 = 16, \quad S_4 = 64, \quad S_8 = 256,$$

so the GP ratio is

$$\boxed{r = 4}.$$

Finally,

$$S_n = \frac{n}{2}(8 + 8(n-1)) = \frac{n}{2}(8n) = \boxed{4n^2}.$$

Doubling the index multiplies S_n by 4, so $S_{16} = 4S_8 = \boxed{1024}$.

Key insight. This is the kind of question where the notation matters. u_n is arithmetic, but S_2, S_4, S_8 are being treated as a *different* three-term geometric sequence.

4 Geometric sequences: the same multiplier is used

4.1 The geometric sequence formulas

Suppose the first term is u_1 and the common ratio is r . Then

$$u_n = u_1 r^{n-1}.$$

For the first n terms,

$$S_n = u_1 \frac{1 - r^n}{1 - r} \quad (r \neq 1).$$

If $|r| < 1$, the infinite sum exists:

$$S_\infty = \frac{u_1}{1 - r}.$$

Key insight. A ratio such as $r = 1.08$ means an 8% increase each step; $r = 0.92$ means an 8% decrease. Translate percentage change into a multiplier first.

4.2 Consecutive terms in a geometric sequence

Again use one notation from start to finish. Let

$$a, b, c$$

be three consecutive *non-zero* terms of a geometric sequence. The common ratio is the same on both steps:

$$r = \frac{b}{a}, \quad r = \frac{c}{b}.$$

Therefore

$$\left[r = \frac{b}{a}, r = \frac{c}{b} \right] \implies \frac{b}{a} = \frac{c}{b} \implies [b^2 = ac].$$

So the square of the middle term equals the product of the two outer terms. In indexed notation,

$$u_n^2 = u_{n-1} u_{n+1}.$$

If the three terms are positive, then $b = \sqrt{ac}$, which is why b is called the geometric mean of a and c .

The exact GP test

Whenever three non-zero expressions are claimed to be consecutive GP terms in the stated order, label them a, b, c and test

$$[b^2 = ac].$$

If the order can be reversed, remember that reciprocal ratios may describe the same three positive values in reverse order.

Worked Example 11 • consecutive GP terms

NO GDC

Three consecutive positive terms of a geometric sequence have sum 28, and the middle term is 8. Find the three terms and all possible common ratios.

Detailed solution

Let the terms be

$$p, \quad 8, \quad q.$$

Because they are consecutive geometric terms,

$$8^2 = pq = 64.$$

Their sum gives

$$p + q = 20.$$

So p and q are the two roots of

$$t^2 - 20t + 64 = 0 = (t - 4)(t - 16).$$

Thus the outside terms are 4 and 16.

There are two possible orders:

$$4, 8, 16 \Rightarrow r = 2,$$

or

$$16, 8, 4 \Rightarrow r = \frac{1}{2}.$$

Answer: $(4, 8, 16)$ with $r = 2$, or $(16, 8, 4)$ with $r = \frac{1}{2}$.

Quick practice - consecutive GP terms

Three consecutive positive GP terms are $p, 12, q$, and $p + q = 30$. Find the possible ordered triples.

Answer. $pq = 12^2 = 144$ and $p + q = 30$, so $\{p, q\} = \{6, 24\}$. The triples are $(6, 12, 24)$ and $(24, 12, 6)$.

Worked Example 12 • find terms and an index

NO GDC

A geometric sequence has $u_1 = 5$ and $r = 3$.

(a) Find u_6 .

(b) Find n if $u_n = 10935$.

(c) Find S_6 .

Detailed solution

(a)

$$u_6 = 5(3)^5 = 1215.$$

(b)

$$10935 = 5(3)^{n-1} \Rightarrow 2187 = 3^{n-1} = 3^7.$$

So $n - 1 = 7$ and $n = 8$.

(c)

$$S_6 = 5 \frac{1 - 3^6}{1 - 3} = 5 \frac{-728}{-2} = 1820.$$

Answer: $u_6 = 1215$, $n = 8$, $S_6 = 1820$.

Worked Example 13 • two known terms

NO GDC

A geometric sequence satisfies

$$u_3 = 12, \quad u_6 = 96,$$

and all terms are positive. Find u_1 and r .

Detailed solution

Write the two conditions:

$$u_1 r^2 = 12,$$

$$u_1 r^5 = 96.$$

Divide the second equation by the first. This is much cleaner than trying to solve both at once:

$$r^3 = \frac{96}{12} = 8.$$

Because the terms are positive,

$$r = 2.$$

Then

$$u_1(2)^2 = 12 \Rightarrow u_1 = 3.$$

Answer: $u_1 = 3, r = 2$.

Exam trick. For a GP, dividing two term equations eliminates u_1 . For an AP, subtracting two term equations eliminates u_1 .

Worked Example 14 • a percentage model

NO GDC

A company sells 40 units in its first month. Sales are modelled as increasing by 10% each month.

- Write down u_1 and r .
- Find the modelled sales in month 12.
- Explain why the answer from the model may not be a whole number.

Context inspired by the recurring IB monthly-sales GP structure, including 24M.1.SL.TZ1.3; data and wording are reconstructed.

Detailed solution

A 10% increase means we keep 100% of the old amount and add 10% more. So the multiplier is

$$r = 1.10.$$

Therefore

$$u_1 = 40, \quad u_{12} = 40(1.10)^{11} \approx 114.125.$$

The model predicts about 114 units. The formula gives a decimal because it is a mathematical model of growth; the real-world count must eventually be interpreted as a whole number.

Answer: Month 12 model value ≈ 114.1 , so about 114 units.

4.3 Can an infinite sum really have a finite answer?

Start with a picture, not a formula

The phrase “**sum to infinity**” sounds impossible at first. How can we add infinitely many numbers and still get an ordinary finite answer?

The secret is that **infinite describes how many pieces there are, not how large the total must be**. If the new pieces become tiny quickly enough, they can fill a fixed amount more and more closely without ever pushing the total past it.

For a geometric series, the question is wonderfully simple:

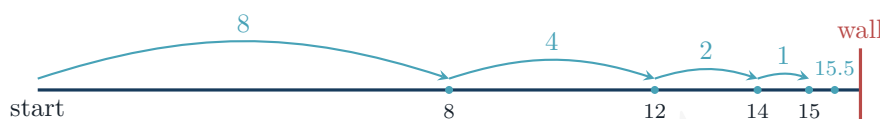
Does the size of each new term shrink?

That happens exactly when $|r| < 1$.

Story Example A • the robot that keeps moving halfway

A robot is 16 m from a wall. On its first move it travels half the distance to the wall, so it moves 8 m. On every later move it again travels half of the distance that is still left.

How far will the robot travel altogether if this process continues forever?



Detailed solution

The distances travelled form the geometric series

$$8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

with first term $u_1 = 8$ and common ratio $r = \frac{1}{2}$.

Before using any infinity formula, we check the ratio:

$$\left| \frac{1}{2} \right| < 1.$$

So every new move is smaller than the previous one. In fact, after many moves the added distance becomes almost invisible.

Look at the partial sums:

$$8, \quad 12, \quad 14, \quad 15, \quad 15.5, \quad 15.75, \dots$$

They are not jumping around randomly. They are creeping closer and closer to 16.

Using the finite sum first,

$$S_n = 8 \frac{1 - (1/2)^n}{1 - 1/2} = 16 \left(1 - \left(\frac{1}{2} \right)^n \right).$$

As n gets larger, $\left(\frac{1}{2}\right)^n$ gets closer and closer to 0. Therefore

$$S_n \rightarrow 16.$$

So we write

$$S_\infty = 16 \text{ m}.$$

The robot never needs one final “magic move”. The total simply gets as close to 16 m as we want.

The child-friendly way to remember it

Imagine filling a box with smaller and smaller pieces. You may use infinitely many pieces, but if every new piece is only a fixed fraction of the previous one, the pieces can fit inside a finite box. That is what convergence means.

4.4 Why $|r| < 1$ is the whole story for a geometric series

From the finite formula to the infinite formula

For any geometric series with first term u_1 and ratio $r \neq 1$, the first n terms add to

$$S_n = u_1 \frac{1 - r^n}{1 - r}.$$

Now imagine increasing n : 10, 100, 1000, ... The only moving part in this formula is r^n .

If $|r| < 1$, repeated multiplication makes $|r|^n$ smaller and smaller, so

$$r^n \longrightarrow 0.$$

That is why the finite formula settles into

$$S_\infty = \frac{u_1}{1 - r}.$$

The formula is not a trick to memorise. It is simply the finite sum formula after the disappearing term r^n has become arbitrarily close to zero.

Story Example B • the prize that doubles every round

A game gives 2 points in round 1, then doubles the prize every round:

$$2 + 4 + 8 + 16 + 32 + \dots$$

A student says, "It is geometric, so I will use $S_\infty = \frac{u_1}{1-r}$." Explain why this is wrong.

Detailed solution

Yes, the series is geometric. But **being geometric is not enough**. Its common ratio is

$$r = \frac{4}{2} = 2.$$

Now check the condition before touching the infinity formula:

$$|r| = 2 > 1.$$

The terms are not shrinking; they are getting larger. The partial sums are

$$2, \quad 6, \quad 14, \quad 30, \quad 62, \quad 126, \dots$$

Each total is racing farther away from any fixed number. There is no value for the partial sums to settle towards. So the correct conclusion is

the series does not have a finite sum to infinity.

If we blindly substituted $u_1 = 2$ and $r = 2$ into $\frac{u_1}{1-r}$, we would get -2 . That answer is obviously impossible for a sum of positive numbers. The mistake is not algebra: the mistake is using a formula when its condition $|r| < 1$ is false.

Three ways a geometric series can fail to settle

- $r = 1$: $5 + 5 + 5 + \dots$ keeps growing.
- $|r| > 1$: terms grow in size, for example $2 + 4 + 8 + \dots$.
- $r = -1$: $6 - 6 + 6 - 6 + \dots$ has partial sums $6, 0, 6, 0, \dots$, so it oscillates instead of settling.

In all three cases, $|r| \geq 1$, so there is no finite geometric sum to infinity.

The exam decision rule - say it out loud

Step 1: Find r . **Step 2:** Check $|r| < 1$. **Step 3:** Only then use $S_\infty = \frac{u_1}{1-r}$.

A useful sentence in a written solution is: "Since $|r| < 1$, the geometric series converges, so the sum to infinity exists." If $|r| \geq 1$, write that it does not converge and stop.

Key insight. For a geometric series, convergence means the partial sums approach one fixed number. When $|r| < 1$, even an alternating series such as $10 - 5 + 2.5 - 1.25 + \dots$ can converge: the totals may zig-zag, but the zig-zags become smaller and smaller.

5 Geometric sums and thresholds

Worked Example 15 • finite geometric sum

NO GDC

A geometric sequence has $u_1 = 800$ and $r = 0.75$. Find the sum of the first 8 terms.

Detailed solution

The terms are shrinking, but the question asks for only the first eight terms, so we use the *finite* formula:

$$S_8 = 800 \frac{1 - 0.75^8}{1 - 0.75}.$$

$$S_8 \approx 2879.75.$$

Answer: $S_8 \approx 2879.75$.

Worked Example 16 • infinite sum

NO GDC

An infinite geometric series begins

$$18 + 12 + 8 + \dots$$

Find its sum to infinity.

Detailed solution

First identify the ratio:

$$r = \frac{12}{18} = \frac{2}{3}.$$

Because

$$\left| \frac{2}{3} \right| < 1,$$

the infinite sum exists. Now we can use

$$S_\infty = \frac{u_1}{1 - r} = \frac{18}{1 - 2/3} = 54.$$

Answer: $S_\infty = 54$.

Common trap. Never use S_∞ before checking $|r| < 1$. If the terms do not shrink to zero, the infinite total cannot settle to a finite number.

Worked Example 17 • least integer index

GDC

A geometric sequence is defined by

$$u_n = 7(1.18)^{n-1}.$$

Find the least integer n for which $u_n > 100$.

Detailed solution

The important word is **least**. We need the first integer index that crosses the threshold.
A GDC table with step 1 gives values around the crossing:

$$u_{17} \approx 98.9, \quad u_{18} \approx 116.7.$$

Therefore the first term above 100 is term 18.

We can also solve the exponential inequality using logarithms, but the final answer must still be an integer sequence index.

Answer: $n = 18$.

A reliable threshold routine

1. Build the formula for u_n or S_n . 2. Use a table with integer step 1. 3. Find the first row that satisfies the inequality. 4. Quote the two neighbouring values to prove that your integer is the first one.

Worked Example 18 • recover r from an infinite sum

NO GDC

A convergent geometric series has first term $u_1 = 12$ and sum to infinity 20.

- (a) Find r .
- (b) Find u_5 .
- (c) Find the sum from term 5 onwards.

Detailed solution

(a)

$$20 = \frac{12}{1 - r}.$$

So

$$20(1 - r) = 12 \Rightarrow 20r = 8 \Rightarrow r = 0.4.$$

(b)

$$u_5 = 12(0.4)^4 = 0.3072.$$

(c) From term 5 onwards, the first term of the remaining geometric series is u_5 and the ratio is still 0.4:

$$\text{tail} = \frac{0.3072}{1 - 0.4} = 0.512.$$

Answer: $r = 0.4$, $u_5 = 0.3072$, **tail** = 0.512.

Sigma notation: turning a long sum into one mathematical sentence

Imagine that a teacher writes fifty terms on the board and asks you to add them. The arithmetic may be easy, but writing every term is slow and one missing term ruins the answer. **Sigma notation** is a compact instruction for exactly the same sum.

The symbol Σ is the capital Greek letter sigma. In this context it simply means: **generate the required terms and add them**.

Formal definition

Let $m, n \in \mathbb{Z}$ with $m \leq n$. Then

$$\sum_{k=m}^n f(k) = f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

The letter k is the **index of summation**. Think of it as a counter. Start at $k = m$, increase by 1, substitute each value into $f(k)$, and stop after $k = n$.

For example,

$$\begin{aligned} \sum_{k=2}^5 (3k+1) &= (3 \cdot 2 + 1) + (3 \cdot 3 + 1) + (3 \cdot 4 + 1) + (3 \cdot 5 + 1) \\ &= 7 + 10 + 13 + 16. \end{aligned}$$

So the reading process is

limits $\longrightarrow$ substitute $\longrightarrow$ generate terms $\longrightarrow$ add.

How many terms?

From $k = m$ to $k = n$, inclusive, there are

$$n - m + 1$$

terms. Thus $k = 4, 5, \dots, 10$ gives $10 - 4 + 1 = 7$ terms. The $+1$ appears because both endpoints are included.

Connection with the notation u_n and S_n

If (u_n) is any sequence, then

$$\sum_{k=1}^n u_k = u_1 + u_2 + \cdots + u_n = S_n.$$

Thus sigma notation is not a new kind of sequence. It is another language for writing a sum of sequence terms. The index letter is a **dummy variable**:

$$\sum_{k=1}^4 (2k+3) = \sum_{j=1}^4 (2j+3).$$

Only the values generated by the rule matter; changing the letter consistently does not change the sum.

The key misconception

The symbol Σ does *not* mean “arithmetic” or “geometric”. It only means “add”. Expand the first few terms first. Constant difference $\Rightarrow$ AP; constant ratio $\Rightarrow$ GP.

Worked Example A: sigma notation hiding an arithmetic series

A small theatre is built row by row. The first row has 7 seats, and each new row has 4 more seats than the previous row. The number of seats in row k is

$$u_k = 7 + 4(k - 1).$$

With n rows, the total number of seats is

$$\sum_{k=1}^n [7 + 4(k - 1)].$$

Detailed solution

Step 1: generate terms from the sigma notation.

k	$7 + 4(k - 1)$	
1	7	
2	11	$\Rightarrow 7 + 11 + 15 + 19 + \dots$
3	15	
4	19	

Step 2: recognise the structure.

$$11 - 7 = 4, \quad 15 - 11 = 4, \quad 19 - 15 = 4.$$

The difference is constant, so the generated terms form an AP with

$$u_1 = 7, \quad d = 4.$$

Its n th term is

$$u_n = u_1 + (n - 1)d = 7 + 4(n - 1) = 4n + 3.$$

Step 3: count the terms. The index runs from 1 to n , so there are exactly n terms.

Step 4: use the arithmetic-series formula.

$$\begin{aligned} \sum_{k=1}^n [7 + 4(k - 1)] &= \frac{n}{2} [2u_1 + (n - 1)d] \\ &= \frac{n}{2} [14 + 4(n - 1)] \\ &= \frac{n}{2} (4n + 10) \\ &= \boxed{n(2n + 5)}. \end{aligned}$$

Why this method is legitimate

There is no special “sigma formula” being used here. The sigma notation generated the familiar AP

$$7, 11, 15, 19, \dots,$$

and that is why the ordinary AP sum formula applies.

Quick check with $n = 4$.

$$7 + 11 + 15 + 19 = 52,$$

while

$$4(2 \cdot 4 + 5) = 52.$$

The direct sum and the formula agree.

IB insight. A term rule that is linear in the index, such as $ak + b$, generates a constant first difference. That is a strong signal that an arithmetic-series method is hidden inside the sigma notation.

Worked Example B: sigma notation hiding a geometric series

Now imagine a game that awards 20 points in round 1, then half as many points in every later round. The score earned in round k is

$$u_k = 20 \left(\frac{1}{2} \right)^{k-1}.$$

After n rounds, the total score is

$$\sum_{k=1}^n 20 \left(\frac{1}{2} \right)^{k-1}.$$

The exponent is $k - 1$, not k , because the first term must use exponent 0:

$$20 \left(\frac{1}{2} \right)^0 = 20.$$

Detailed solution

Step 1: generate the first few terms.

$$20, \quad 10, \quad 5, \quad \frac{5}{2}, \quad \dots$$

Step 2: recognise the structure.

$$\frac{10}{20} = \frac{1}{2}, \quad \frac{5}{10} = \frac{1}{2}, \quad \frac{(5/2)}{5} = \frac{1}{2}.$$

The ratio is constant, so the terms form a GP with

$$u_1 = 20, \quad r = \frac{1}{2}.$$

Step 3: count the terms. Since $k = 1, 2, \dots, n$, there are n terms.

Step 4: use the finite geometric-series formula.

$$\begin{aligned} \sum_{k=1}^n 20 \left(\frac{1}{2} \right)^{k-1} &= u_1 \frac{1 - r^n}{1 - r} \\ &= 20 \frac{1 - (1/2)^n}{1 - 1/2} \\ &= \boxed{40(1 - 2^{-n})}. \end{aligned}$$

Why this connects naturally to sum to infinity

Because $\left| \frac{1}{2} \right| < 1$, we have $2^{-n} \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$40(1 - 2^{-n}) \rightarrow 40,$$

which agrees with

$$S_{\infty} = \frac{u_1}{1 - r} = \frac{20}{1 - 1/2} = 40.$$

The terms continue forever, but they shrink so quickly that the running total approaches a fixed number.

Quick check with $n = 4$.

$$20 + 10 + 5 + \frac{5}{2} = \frac{75}{2},$$

and

$$40(1 - 2^{-4}) = 40 \left(1 - \frac{1}{16} \right) = \frac{75}{2}.$$

Again the two descriptions match exactly.

Sigma notation: the exam method

Sigma notation should now feel like a language for writing sums rather than a new formula to memorise. The safest approach is always to **translate first and calculate second**.

A six-step method

For

$$\sum_{k=m}^n f(k),$$

ask the following questions in order:

1. **Where does the index start and stop?** Read the lower and upper limits carefully.
2. **How many terms are there?** Use $n - m + 1$ when the limits are m and n .
3. **What are the first three terms?** Substitute the first three index values into $f(k)$.
4. **What pattern appears?** Constant difference $\Rightarrow$ AP; constant ratio $\Rightarrow$ GP.
5. **Which familiar formula matches?** Rewrite the information using u_1, d or u_1, r .
6. **Can you check a small case?** Choose a small upper limit and compare direct addition with your formula.

Common mistakes

- Starting at $k = 1$ automatically when the lower limit is actually 0, 2, or another number.
- Forgetting that both endpoints are included when counting terms.
- Treating Σ itself as an AP or GP symbol.
- Forgetting that $(1/2)^{k-1}$ starts with exponent 0 when $k = 1$.
- Changing the index letter in one place but not everywhere in the summand.
- Using S_∞ before checking $|r| < 1$.

The picture to keep in your head

index values $\rightarrow$ term rule $\rightarrow$ list of terms $\rightarrow$ recognise AP/GP $\rightarrow$ sum

The sigma symbol does not choose the formula. **You** choose the formula after recognising the pattern.

Exam habit. In a difficult IB question, write the first two or three generated terms even when the pattern seems obvious. That single line of working often prevents an index error and makes the rest of the solution much clearer.

6 Mixed IB-style examples: deciding what to use

Worked Example 19 • arithmetic versus geometric growth

GDC

Two runners both travel 500 m on day 1. Runner C increases the daily distance by 100 m each day. Runner D increases the previous day's distance by 2% each day.

- Find each runner's distance on day 20.
- Find the first day on which Runner D travels farther than Runner C.

Structure adapted from IB reference 21M.1.SL.TZ1.8; names and wording are reconstructed.

Detailed solution

Runner C changes by addition, so this is arithmetic:

$$C_n = 500 + 100(n - 1).$$

Runner D changes by multiplication, so this is geometric:

$$D_n = 500(1.02)^{n-1}.$$

At $n = 20$,

$$C_{20} = 2400, \quad D_{20} \approx 728.4.$$

For the crossover, use a GDC table for both expressions and scan integer n until $D_n > C_n$ for the first time. The first such day is

$$n = 185.$$

The key skill is not the calculator. It is recognising that the two runners follow *different sequence types* before comparing them.

Answer: Day 20: 2400 m and 728.4 m; first crossover at day 185.

Worked Example 20 • simultaneous conditions in an AP

NO GDC

The sum of the first three terms of an arithmetic sequence is 21, and the sixth term is 18. Find u_1 and d .

Detailed solution

Write each piece of information as an equation.

The first three terms are

$$u_1, \quad u_1 + d, \quad u_1 + 2d.$$

Their sum is 21:

$$3u_1 + 3d = 21 \Rightarrow u_1 + d = 7. \quad (1)$$

The sixth term is 18:

$$u_1 + 5d = 18. \quad (2)$$

Subtract (1) from (2):

$$4d = 11 \Rightarrow d = \frac{11}{4}.$$

Then

$$u_1 = 7 - \frac{11}{4} = \frac{17}{4}.$$

$$\textbf{Answer: } u_1 = \frac{17}{4}, \quad d = \frac{11}{4}.$$

Worked Example 21 • one equation links an AP and a GP

NO GDC

An arithmetic sequence and a geometric sequence both have first term 4 and second term 10.

- Find the common difference of the AP and the common ratio of the GP.
- Find the fifth term of each sequence.
- Which sequence is larger at term 5, and why is that not surprising?

Detailed solution

For the AP,

$$d = 10 - 4 = 6.$$

For the GP,

$$r = \frac{10}{4} = 2.5.$$

Now

$$A_5 = 4 + 4(6) = 28,$$

while

$$G_5 = 4(2.5)^4 = 156.25.$$

The GP is already much larger because repeated multiplication eventually grows much faster than repeated addition when $r > 1$.

$$\textbf{Answer: } d = 6, \quad r = 2.5, \quad A_5 = 28, \quad G_5 = 156.25.$$

7 Past-paper patterns and harder CF Maths examples

How these examples are used

The next examples imitate the multi-part progression of IB questions: one early result is deliberately reused later. They are not copies of official wording. The reference codes are included so students can find the original style in past-paper collections.

Worked Example 22 • past-paper pattern: explicit AP

NO GDC

An arithmetic sequence is defined by

$$u_n = 18 - 4n.$$

- (a) Find u_1 and d .
- (b) Find n if $u_n = -42$.
- (c) Find S_n for that value of n .

Adapted from the structure of IB reference 22M.1.SL.TZ2.2.

Detailed solution

(a)

$$u_1 = 18 - 4 = 14.$$

Also

$$u_{n+1} - u_n = [18 - 4(n+1)] - (18 - 4n) = -4,$$

so $d = -4$.

(b)

$$18 - 4n = -42 \Rightarrow -4n = -60 \Rightarrow n = 15.$$

(c)

$$S_{15} = \frac{15}{2}(u_1 + u_{15}) = \frac{15}{2}(14 - 42) = -210.$$

Answer: $u_1 = 14$, $d = -4$, $n = 15$, $S_{15} = -210$.

Worked Example 23 • difficult CF Maths: partial sums create an AP and a hidden GP

NO GDC

A sequence has partial sums

$$S_n = n^2 + pn,$$

where p is a real constant. The terms u_2, u_5, u_9 , in that order, are consecutive terms of a geometric sequence.

- (a) Find u_n in terms of n and p .
- (b) Show that (u_n) is arithmetic.
- (c) Find p .
- (d) Find the common ratio of the geometric sequence u_2, u_5, u_9 .
- (e) Find $u_{10} + u_{11} + \cdots + u_{20}$.

Detailed solution

(a) For $n \geq 2$,

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= n^2 + pn - [(n-1)^2 + p(n-1)] \\ &= 2n + p - 1. \end{aligned}$$

This also agrees with $u_1 = S_1 = p + 1$.

(b)

$$u_{n+1} - u_n = 2,$$

so the sequence is arithmetic with common difference 2.

(c)

$$u_2 = p + 3, \quad u_5 = p + 9, \quad u_9 = p + 17.$$

Because these are consecutive GP terms,

$$(p + 9)^2 = (p + 3)(p + 17).$$

Hence

$$p^2 + 18p + 81 = p^2 + 20p + 51 \Rightarrow p = 15.$$

(d) The selected terms are

$$18, 24, 32,$$

so

$$r = \frac{24}{18} = \frac{4}{3}.$$

(e) Use partial sums rather than adding eleven terms:

$$u_{10} + \cdots + u_{20} = S_{20} - S_9.$$

With $p = 15$,

$$S_{20} = 700, \quad S_9 = 216,$$

so the required sum is

$$484.$$

Answer: $u_n = 2n + p - 1$, $p = 15$, **GP ratio** $\frac{4}{3}$, **required sum** = 484.

Worked Example 24 • difficult CF Maths: selected AP terms form a GP

GDC

An arithmetic sequence has non-zero common difference d . The terms u_2, u_4, u_7 , in that order, are consecutive terms of a geometric sequence. It is also known that

$$S_6 = 66.$$

(a) Show that $u_1 = 3d$.

(b) Hence find u_1 and d .

(c) Find the common ratio of the geometric sequence u_2, u_4, u_7 and its next term.

(d) Find a formula for S_n of the arithmetic sequence.

(e) Find the least positive integer n for which $S_n > 1000$, and verify it using neighbouring values.

Detailed solution

(a)

$$u_2 = u_1 + d, \quad u_4 = u_1 + 3d, \quad u_7 = u_1 + 6d.$$

The GP condition gives

$$(u_1 + 3d)^2 = (u_1 + d)(u_1 + 6d).$$

After expanding and cancelling u_1^2 ,

$$3d^2 = u_1d.$$

Since $d \neq 0$,

$$u_1 = 3d.$$

(b)

$$66 = S_6 = \frac{6}{2}(2u_1 + 5d) = 3(2u_1 + 5d).$$

Thus

$$2u_1 + 5d = 22.$$

Substitute $u_1 = 3d$:

$$11d = 22 \Rightarrow d = 2, \quad u_1 = 6.$$

(c)

$$u_2 = 8, \quad u_4 = 12, \quad u_7 = 18.$$

Therefore

$$r = \frac{12}{8} = \frac{3}{2},$$

and the next GP term is

$$18 \left(\frac{3}{2} \right) = 27.$$

(d)

$$S_n = \frac{n}{2}(12 + 2(n-1)) = n(n+5).$$

(e) A GDC table gives the first crossing at $n = 30$. Verify:

$$S_{29} = 986 < 1000, \quad S_{30} = 1050 > 1000.$$

Answer: $u_1 = 6$, $d = 2$, **GP ratio** $\frac{3}{2}$, **next GP term** 27, $S_n = n(n+5)$, **least** $n = 30$.

Worked Example 25 • difficult CF Maths: two geometric sums determine the whole GP

NO GDC

A geometric sequence has positive first term and satisfies

$$S_2 = 9, \quad S_4 = 45.$$

(a) Show that $S_4 = S_2(1 + r^2)$.(b) Find r and u_1 .(c) Find u_8 .(d) Find the least positive integer n for which $S_n > 3000$, verifying the two neighbouring partial sums.

Detailed solution

(a) Group the first four terms into two pairs:

$$\begin{aligned} S_4 &= (u_1 + u_1r) + (u_1r^2 + u_1r^3) \\ &= u_1(1 + r)(1 + r^2). \end{aligned}$$

Since $S_2 = u_1(1 + r)$,

$$S_4 = S_2(1 + r^2).$$

(b)

$$45 = 9(1 + r^2) \Rightarrow r^2 = 4.$$

Thus $r = 2$ or $r = -2$. If $r = -2$, then $9 = u_1(1 - 2) = -u_1$, making $u_1 = -9$, which contradicts the positive-first-term condition. Hence

$$r = 2.$$

Then

$$9 = u_1(1 + 2) \Rightarrow u_1 = 3.$$

(c)

$$u_8 = 3(2)^7 = 384.$$

(d)

$$S_n = 3 \frac{1 - 2^n}{1 - 2} = 3(2^n - 1).$$

The first value above 3000 occurs at $n = 10$, and

$$S_9 = 1533 < 3000, \quad S_{10} = 3069 > 3000.$$

Answer: $r = 2$, $u_1 = 3$, $u_8 = 384$, least $n = 10$.

Worked Example 26 • difficult CF Maths: term crossover versus total crossover GDC

Two models begin at the same time. Model A follows the arithmetic sequence

$$A_n = 80 + 15(n - 1),$$

while Model G follows the geometric sequence

$$G_n = 20(1.18)^{n-1}.$$

- Find the first positive integer n for which $G_n > A_n$.
- Write formulas for the cumulative totals $S_n^{(A)}$ and $S_n^{(G)}$.
- Find the first positive integer n for which $S_n^{(G)} > S_n^{(A)}$.
- Explain why the answer in part (c) is later than the answer in part (a).

Detailed solution

(a) Use a GDC table because one expression is linear in n and the other is exponential. The first crossing is at $n = 19$:

$$G_{18} \approx 333.445 < 335 = A_{18},$$

$$G_{19} \approx 393.465 > 350 = A_{19}.$$

(b)

$$S_n^{(A)} = \frac{n}{2}(160 + 15(n - 1)) = \frac{n}{2}(15n + 145),$$

while

$$S_n^{(G)} = 20 \frac{1 - 1.18^n}{1 - 1.18} = \frac{1000}{9}(1.18^n - 1).$$

(c) Comparing the two cumulative formulas in a GDC table gives the first total crossover at $n = 25$:

$$S_{24}^{(G)} \approx 5789.890 < 6060 = S_{24}^{(A)},$$

$$S_{25}^{(G)} \approx 6852.070 > 6500 = S_{25}^{(A)}.$$

(d) The geometric individual term becomes larger at $n = 19$, but Model A has already built up a large cumulative lead during the earlier stages. Model G therefore needs several more stages to recover that earlier deficit.

Answer: First term crossover $n = 19$; first cumulative crossover $n = 25$.

HL MATERIAL • linked arithmetic and geometric sequences

A recurring AA HL pattern asks students to compare an arithmetic sequence and a geometric sequence that share one or more terms, then reuse those relationships in later algebra or logarithms. IB reference 24M.1.AHL.TZ2.10 is a useful historic example of that style. The key approach is always the same: write the AP condition using equal differences, write the GP condition using equal ratios, and solve only after both structures are visible.

8 Final strategy sheet

General sequence and partial-sum links

$$S_n = u_1 + u_2 + \cdots + u_n, \quad \boxed{u_1 = S_1}, \quad \boxed{u_n = S_n - S_{n-1}} \quad (n \geq 2).$$

For a block of terms,

$$\boxed{u_p + u_{p+1} + \cdots + u_q = S_q - S_{p-1}}.$$

Trigger: if the question gives S_n and asks for a term, subtract consecutive partial sums.

Arithmetic sequences and series

$$\boxed{u_n = u_1 + (n-1)d}, \quad \boxed{S_n = \frac{n}{2}[2u_1 + (n-1)d] = \frac{n}{2}(u_1 + u_n)}.$$

For three consecutive AP terms a, b, c :

$$\boxed{d = b - a}, \quad \boxed{d = c - b} \Rightarrow b - a = c - b \Rightarrow \boxed{2b = a + c} \Rightarrow \boxed{b = \frac{a + c}{2}}.$$

Fast habits: subtract two term equations to eliminate u_1 ; for decreasing APs, a maximum partial sum occurs around the sign change of the terms.

Geometric sequences and series

$$\boxed{u_n = u_1 r^{n-1}}, \quad \boxed{S_n = u_1 \frac{1 - r^n}{1 - r}}, \quad \boxed{S_\infty = \frac{u_1}{1 - r}} \quad (|r| < 1).$$

For three consecutive non-zero GP terms a, b, c :

$$\boxed{r = \frac{b}{a}}, \quad \boxed{r = \frac{c}{b}} \Rightarrow \boxed{b^2 = ac}.$$

The same rule applies when the “terms” are expressions such as partial sums:

$$\boxed{S_j^2 = S_i S_k} \quad \text{if } S_i, S_j, S_k \text{ are consecutive GP terms.}$$

Final decision checklist

1. Decide whether you are working with a **term** u_n , a **sum** S_n , or a new sequence built from selected terms/sums.
2. Identify the structure: additive $\Rightarrow$ AP; multiplicative $\Rightarrow$ GP.
3. Keep notation consistent. For three consecutive terms use a, b, c in their stated order.
4. AP consecutive condition: $2b = a + c$. GP consecutive condition: $b^2 = ac$.
5. If AP partial sums form a GP, first express every required S_n in the *same AP variables*, then apply the GP condition.
6. If an even power of r appears, check both signs unless positivity/order removes one.
7. For GDC threshold questions, verify the neighbouring integer indices; a decimal intersection is not the final answer.
8. Reuse earlier subparts. In IB questions, a “show that” result is usually designed for the next part.

Pattern first $\rightarrow$ notation second $\rightarrow$ formula third $\rightarrow$ check at the end.

Three questions to ask before every sequence problem

1. Is the pattern additive, multiplicative, or neither?
2. Is the question about one term or about a sum?
3. Is the index required to be an integer, and do I actually need a GDC?

Past-paper style references used

Reference	Main structure used in this chapter
21M.1.SL.TZ1.3	A condition linking one arithmetic term and the corresponding partial sum.
21M.2.SL.TZ2.3	Descending arithmetic sequence and maximum partial sum.
22M.1.SL.TZ2.2	Reading u_1 , d and an index from an explicit arithmetic rule.
21M.1.SL.TZ1.8	Comparing arithmetic and geometric growth and finding the first crossover index.
24M.1.AHL.TZ2.10	Linked AP/GP structure in a longer HL progression.