

Sequences Homework - Worked Solutions

Sequences, Arithmetic and Geometric Sequences

Full derivations • Exact answers • IB-style communication

These solutions are written to model the level of algebraic communication expected in IB Mathematics AA. Where a question admits more than one real solution, all valid cases are retained.

Source note.

Solutions 1-22 correspond to teacher-selected core practice. Solutions 23-35 correspond to paraphrased/adapted IB past-paper questions, with the original references shown for lookup. Solutions 36-41 are original CF Maths combined AP/GP synthesis. First-term notation is standardised as u_1 (or v_1 for a second sequence).

Section A - Foundations and Core Technique

1. Reading terms and a partial sum

From the listed sequence,

$$u_2 = 9, \quad u_6 = 19.$$

Also,

$$S_4 = 5 + 9 + 11 + 12 = 37.$$

Answer: $u_2 = 9, u_6 = 19, S_4 = 37$

2. Consecutive arithmetic terms

For three consecutive arithmetic terms, twice the middle term equals the sum of the two outer terms:

$$2(k - 2) = 3k + (k + 7).$$

Hence

$$2k - 4 = 4k + 7 \implies -11 = 2k \implies k = -\frac{11}{2}.$$

Answer: $k = -\frac{11}{2}$

3. Explicit geometric sequence

(a) We compare consecutive terms:

$$\frac{u_{n+1}}{u_n} = \frac{6(1/2)^n}{6(1/2)^{n-1}} = \frac{1}{2},$$

which is constant. Therefore the sequence is geometric.

(b) At $n = 1$,

$$u_1 = 6 \left(\frac{1}{2}\right)^0 = 6,$$

and the common ratio is

$$r = \frac{1}{2}.$$

(c)

$$u_{16} = 6 \left(\frac{1}{2}\right)^{15} = \frac{3}{16384} \approx 1.83105 \times 10^{-4}.$$

To three significant figures,

$$u_{16} \approx 0.000183.$$

Answer: $u_1 = 6, r = \frac{1}{2}, u_{16} \approx 0.000183$

4. Recovering a GP from two non-consecutive terms

Let the first term be a and common ratio be r . Then

$$ar^5 = \frac{16}{3}, \quad ar^9 = \frac{256}{3}.$$

Dividing,

$$r^4 = \frac{256/3}{16/3} = 16.$$

Thus

$$r = 2 \quad \text{or} \quad r = -2.$$

If $r = 2$,

$$a = \frac{16/3}{2^5} = \frac{1}{6}.$$

If $r = -2$,

$$a = \frac{16/3}{(-2)^5} = -\frac{1}{6}.$$

Therefore the two possible real sequences are

$$\boxed{u_n = \frac{1}{6} 2^{n-1}} \quad \text{or} \quad \boxed{u_n = -\frac{1}{6} (-2)^{n-1}}.$$

5. Arithmetic-series sums

(a) The first 50 positive multiples of 11 are

$$11, 22, \dots, 550.$$

Therefore

$$S = 11(1 + 2 + \dots + 50) = 11 \cdot \frac{50 \cdot 51}{2} = 14025.$$

(b) The positive multiples of 7 below 1000 end at

$$7 \cdot 142 = 994.$$

Hence

$$S = 7(1 + 2 + \dots + 142) = 7 \cdot \frac{142 \cdot 143}{2} = 71071.$$

Including 0 does not change the sum.

(c)

$$1 + 2 + \dots + 100 = 5050.$$

The multiples of 3 are $3, 6, \dots, 99 = 3 \cdot 33$, with sum

$$3(1 + 2 + \dots + 33) = 3 \cdot \frac{33 \cdot 34}{2} = 1683.$$

So the required sum is

$$5050 - 1683 = 3367.$$

(d) Numbers beginning with 4 are $400, 401, \dots, 499$:

$$S_A = \frac{100}{2}(400 + 499) = 44950.$$

Three-digit numbers ending in 4 are $104, 114, \dots, 994$; there are 90:

$$S_B = \frac{90}{2}(104 + 994) = 49410.$$

Those counted twice are $404, 414, \dots, 494$; there are 10:

$$S_{A \cap B} = \frac{10}{2}(404 + 494) = 4490.$$

By inclusion-exclusion,

$$S = 44950 + 49410 - 4490 = 89870.$$

Answer: (a) 14025; (b) 71071; (c) 3367; (d) 89870

6. Finding an AP and then its sum

For consecutive arithmetic terms,

$$2(2k + 3) = (k - 1) + (27 - k).$$

Thus

$$4k + 6 = 26 \implies k = 5.$$

The first three terms are therefore

$$4, 13, 22,$$

so $a = 4$ and $d = 9$. Then

$$S_{25} = \frac{25}{2} (2(4) + 24(9)) = \frac{25}{2} (224) = 2800.$$

Answer: $k = 5$, $S_{25} = 2800$

7. Term and sum conditions in an AP

Let the first term be a and common difference be d . From $u_6 = 21$,

$$a + 5d = 21. \tag{1}$$

Also

$$S_{17} = \frac{17}{2} (2a + 16d) = 0,$$

so

$$a + 8d = 0. \tag{2}$$

Subtracting (1) from (2),

$$3d = -21 \implies d = -7.$$

Then

$$a + 5(-7) = 21 \implies a = 56.$$

Hence the first two terms are

$$56, 49.$$

Answer: 56, 49

8. Recovering an AP from two partial sums

Using

$$S_n = \frac{n}{2} (2a + (n - 1)d),$$

we have

$$S_3 = 3(a + d) = 9 \implies a + d = 3. \tag{1}$$

Also

$$S_6 = 3(2a + 5d) = 90 \implies 2a + 5d = 30. \tag{2}$$

Twice (1) gives $2a + 2d = 6$. Subtracting from (2),

$$3d = 24 \implies d = 8,$$

and hence $a = -5$. Therefore

$$S_{10} = 5(2(-5) + 9(8)) = 5(62) = 310.$$

Answer: $S_{10} = 310$

Section B - Structure, Series and Connections

9. Pairing consecutive terms of an AP

Let

$$u_n = a + (n - 1)d.$$

Define

$$v_n = u_{2n-1} + u_{2n}.$$

Then

$$v_n = [a + (2n - 2)d] + [a + (2n - 1)d] = 2a + (4n - 3)d.$$

Therefore

$$v_{n+1} - v_n = 4d,$$

which is constant. Hence (v_n) is arithmetic.

Now

$$u_2 = a + d = 5, \quad u_{10} = a + 9d = 61.$$

Subtracting,

$$8d = 56 \implies d = 7,$$

so $a = -2$. Thus

$$v_{30} = 2(-2) + (4 \cdot 30 - 3)(7) = -4 + 819 = 815.$$

Answer: The paired sequence is arithmetic with difference $4d$; its 30th term is 815.

10. A rational parameter in an AP

Since the 3rd and 4th terms are $1/k$ and k ,

$$d = k - \frac{1}{k}.$$

The 6th term is two further steps from the 4th, so

$$k^2 + 1 = k + 2 \left(k - \frac{1}{k} \right) = 3k - \frac{2}{k}.$$

Multiplying by k ,

$$k^3 + k = 3k^2 - 2,$$

so

$$k^3 - 3k^2 + k + 2 = 0.$$

Factor:

$$(k - 2)(k^2 - k - 1) = 0.$$

Since $k \in \mathbb{Q}$, the irrational roots of $k^2 - k - 1 = 0$ are rejected. Hence

$$k = 2.$$

Then

$$d = 2 - \frac{1}{2} = \frac{3}{2}.$$

Since $u_3 = 1/2$,

$$u_1 = u_3 - 2d = \frac{1}{2} - 3 = -\frac{5}{2}.$$

Therefore

$$u_n = -\frac{5}{2} + (n-1)\frac{3}{2} = \frac{3n-8}{2}.$$

Answer: $k = 2, \quad u_n = \frac{3n-8}{2}$

11. GP from second term and a three-term sum

Let the first term be a and common ratio be r . Since the second term is 6,

$$ar = 6.$$

The first three terms are $a, 6, 6r$, so

$$a + 6 + 6r = -14 \implies a + 6r = -20.$$

Using $a = 6/r$,

$$\frac{6}{r} + 6r = -20.$$

Multiply by r :

$$6 + 6r^2 = -20r \implies 3r^2 + 10r + 3 = 0.$$

Hence

$$(3r + 1)(r + 3) = 0,$$

so

$$r = -\frac{1}{3} \quad \text{or} \quad r = -3.$$

The fourth term is

$$u_4 = ar^3 = (ar)r^2 = 6r^2.$$

Therefore

$$u_4 = 6\left(\frac{1}{9}\right) = \frac{2}{3} \quad \text{or} \quad u_4 = 6(9) = 54.$$

Answer: $u_4 = \frac{2}{3} \text{ or } 54$

12. An AP and a GP linked by their second term

Let the AP have common difference d . Since its first term is 1, its second term is $1 + d$.
The GP has first term 1 and the same second term, so its common ratio is

$$r = 1 + d.$$

The condition gives

$$1 + 13d = 3(1 + d)^2.$$

Expanding,

$$1 + 13d = 3 + 6d + 3d^2,$$

so

$$3d^2 - 7d + 2 = 0 = (3d - 1)(d - 2).$$

Thus

$$d = 2 \quad \text{or} \quad d = \frac{1}{3}.$$

If $d = 2$, then $r = 3$:

$$\text{AP } u_{20} = 1 + 19(2) = 39, \quad \text{GP } v_{20} = 3^{19}.$$

If $d = 1/3$, then $r = 4/3$:

$$\text{AP } u_{20} = 1 + \frac{19}{3} = \frac{22}{3}, \quad \text{GP } v_{20} = \left(\frac{4}{3}\right)^{19}.$$

$$\text{Answer: } (39, 3^{19}) \text{ or } \left(\frac{22}{3}, (4/3)^{19}\right)$$

13. Infinite and finite sums of the same GP

For a convergent GP,

$$S_3 = S_{\infty}(1 - r^3).$$

Hence

$$21 = \frac{64}{3}(1 - r^3).$$

So

$$1 - r^3 = \frac{63}{64} \implies r^3 = \frac{1}{64} \implies r = \frac{1}{4}.$$

Now

$$S_5 = S_{\infty}(1 - r^5) = \frac{64}{3} \left(1 - \frac{1}{4^5}\right).$$

Thus

$$S_5 = \frac{64}{3} \cdot \frac{1023}{1024} = \frac{341}{16}.$$

$$\text{Answer: } S_5 = \frac{341}{16}$$

14. Bouncing ball and a geometric time series

(a) The first downward journey takes 1 second. The first rebound takes 0.9 seconds upward and 0.9 seconds downward, contributing $2(0.9)$. Each later up-and-down pair is multiplied by another factor of 0.9. Hence

$$1 + 2(0.9) + 2(0.9)^2 + \dots$$

(b) If S_n contains the first n displayed terms, then

$$S_n = 1 + 2 \sum_{k=1}^{n-1} (0.9)^k.$$

Using the finite GP formula,

$$S_n = 1 + 2 \left[\frac{0.9(1 - 0.9^{n-1})}{1 - 0.9} \right] = 1 + 18(1 - 0.9^{n-1}).$$

Therefore

$$\boxed{S_n = 19 - 18(0.9)^{n-1}}.$$

(c) As $n \rightarrow \infty$, $(0.9)^{n-1} \rightarrow 0$, so

$$S_{\infty} = 19.$$

Answer: Total time = 19 seconds

15. Partial sums of an AP forming a GP

Let the original AP have first term a and common difference d . Then

$$S_1 = a,$$

$$S_2 = 2a + d,$$

and

$$S_4 = \frac{4}{2}(2a + 3d) = 4a + 6d.$$

For S_1, S_2, S_4 to form a GP,

$$S_2^2 = S_1 S_4.$$

Thus

$$(2a + d)^2 = a(4a + 6d).$$

Expanding,

$$4a^2 + 4ad + d^2 = 4a^2 + 6ad,$$

so

$$d^2 = 2ad \implies d(d - 2a) = 0.$$

Case 1: $d = 0$. Then

$$S_1, S_2, S_4 = a, 2a, 4a,$$

so the common ratio is 2.

Case 2: $d = 2a$. Then

$$S_1, S_2, S_4 = a, 4a, 16a,$$

so the common ratio is 4.

Answer: Common ratio 2 or 4

16. Three consecutive GP terms from symmetric information

Write the three consecutive GP terms as

$$\frac{y}{r}, \quad y, \quad yr.$$

Then

$$y \left(r + 1 + \frac{1}{r} \right) = \frac{7}{3}. \quad (1)$$

Also

$$y^2 \left(r^2 + 1 + \frac{1}{r^2} \right) = \frac{91}{9}. \quad (2)$$

A convenient alternative is to use the GP identity $y^2 = xz$.

From

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz),$$

we get

$$\frac{49}{9} = \frac{91}{9} + 2(xy + yz + xz),$$

so

$$xy + yz + xz = -\frac{7}{3}.$$

Using $xz = y^2$,

$$y(x + z) + y^2 = -\frac{7}{3}.$$

But $x + z = \frac{7}{3} - y$, so

$$y\left(\frac{7}{3} - y\right) + y^2 = -\frac{7}{3},$$

which simplifies to

$$\frac{7}{3}y = -\frac{7}{3},$$

so

$$y = -1.$$

Then

$$x + z = \frac{10}{3}, \quad xz = y^2 = 1.$$

Hence x and z are the roots of

$$t^2 - \frac{10}{3}t + 1 = 0,$$

which gives

$$t = 3 \quad \text{or} \quad t = \frac{1}{3}.$$

Therefore

$$(x, y, z) = \left(3, -1, \frac{1}{3}\right) \quad \text{or} \quad \left(\frac{1}{3}, -1, 3\right).$$

17. Partial sums forming another AP

For an AP with first term a and common difference d ,

$$S_2 = 2a + d,$$

$$S_5 = \frac{5}{2}(2a + 4d) = 5a + 10d,$$

$$S_7 = \frac{7}{2}(2a + 6d) = 7a + 21d.$$

Since S_2, S_5, S_7 form an AP,

$$S_5 - S_2 = S_7 - S_5.$$

Thus

$$(5a + 10d) - (2a + d) = (7a + 21d) - (5a + 10d),$$

so

$$3a + 9d = 2a + 11d \implies a = 2d.$$

The common difference of the new AP is

$$S_5 - S_2 = 3a + 9d = 3(2d) + 9d = 15d.$$

Answer: $15d$

Section C - Proof, Sigma Notation and Extension

18. Sequence values as rectangle areas

Since

$$u_n = 3 + 2n,$$

we have

$$u_1 = 5, \quad u_2 = 7, \quad u_3 = 9, \quad u_4 = 11.$$

Each rectangle has width 1 and height u_n , so its area is exactly u_n . Therefore the sum of the areas of the first n rectangles is

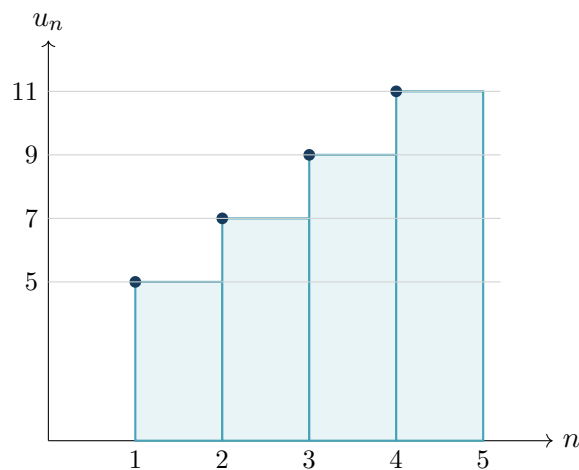
$$u_1 + u_2 + \cdots + u_n = S_n.$$

Also,

$$u_{n+1} = 3 + 2(n+1) = 3 + 2n + 2 = u_n + 2.$$

Finally, when one new rectangle is added to the first n rectangles, its area is u_{n+1} , so

$$S_{n+1} = S_n + u_{n+1}.$$

19. Deriving the formula for $\sum k^2$

(a) By direct expansion,

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1.$$

(b) Rearranging gives

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1.$$

For $x = 0, 1, \dots, n$:

$$1^3 - 0^3 = 3(0)^2 + 3(0) + 1,$$

$$2^3 - 1^3 = 3(1)^2 + 3(1) + 1,$$

$$3^3 - 2^3 = 3(2)^2 + 3(2) + 1,$$

and so on, up to

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1.$$

(c) Adding vertically, the cubic terms telescope:

$$(n+1)^3 = 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + (n+1).$$

(d) Substitute

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} :$$

$$(n+1)^3 = 3 \sum_{k=1}^n k^2 + \frac{3}{2}n(n+1) + (n+1).$$

Hence

$$3 \sum_{k=1}^n k^2 = (n+1)^3 - \frac{3}{2}n(n+1) - (n+1).$$

Factor $(n+1)$:

$$3 \sum_{k=1}^n k^2 = (n+1) \left[(n+1)^2 - \frac{3}{2}n - 1 \right].$$

Simplifying the bracket,

$$(n+1)^2 - \frac{3}{2}n - 1 = n^2 + \frac{1}{2}n = \frac{n(2n+1)}{2}.$$

Therefore

$$\boxed{\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}}.$$

20. Simplifying a quadratic sigma sum

Expand the summand:

$$(k+1)(k+2) = k^2 + 3k + 2.$$

Therefore

$$\sum_{k=1}^n (k+1)(k+2) = \sum k^2 + 3 \sum k + 2 \sum 1.$$

So

$$= \frac{n(n+1)(2n+1)}{6} + 3 \frac{n(n+1)}{2} + 2n.$$

Putting everything over 6 and factoring gives

$$\boxed{\sum_{k=1}^n (k+1)(k+2) = \frac{n(n^2 + 6n + 11)}{3}}.$$

For $n = 10$,

$$\frac{10(100 + 60 + 11)}{3} = \frac{1710}{3} = 570.$$

Answer: $\frac{n(n^2 + 6n + 11)}{3}$; for $n = 10$, 570

21. Reciprocal AP and a right-angled triangle

Since

$$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$$

are consecutive terms of an AP,

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}.$$

Hence

$$\frac{2}{y} = \frac{x+z}{xz} \implies y = \frac{2xz}{x+z}. \quad (1)$$

All three proposed lengths are positive because $x > y > z > 0$.

We show that

$$(x-z)^2 + y^2 = (x-y+z)^2.$$

Expanding the difference between the two sides,

$$(x-z)^2 + y^2 - (x-y+z)^2 = 2y(x+z) - 4xz.$$

Using (1),

$$2y(x+z) - 4xz = 2 \left(\frac{2xz}{x+z} \right) (x+z) - 4xz = 0.$$

Therefore

$$(x-z)^2 + y^2 = (x-y+z)^2.$$

By the converse of Pythagoras, the three positive numbers can be side lengths of a right-angled triangle, with hypotenuse $x - y + z$.

22. A weighted infinite geometric series

Let

$$S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \dots$$

Then

$$\frac{S}{2} = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{4}{64} + \dots$$

Subtract:

$$S - \frac{S}{2} = \frac{1}{4} + \left(\frac{2}{8} - \frac{1}{8} \right) + \left(\frac{3}{16} - \frac{2}{16} \right) + \dots$$

Thus

$$\frac{S}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

The right-hand side is a geometric series with first term $1/4$ and ratio $1/2$, so

$$\frac{S}{2} = \frac{1/4}{1 - 1/2} = \frac{1}{2}.$$

Therefore

$$\boxed{S = 1}.$$

Section D - IB Past-Paper Practice: Worked Solutions

The original IB paper references are shown. These solutions correspond to the paraphrased/adapted questions in the worksheet.

23. IB ref: 21M.1.SL.TZ1.3

Given

$$u_8 = 8,$$

we have

$$a + 7d = 8. \quad (1)$$

Also

$$S_8 = \frac{8}{2}(a + u_8) = 8,$$

so

$$4(a + 8) = 8 \implies a = -6.$$

Substitute into (1):

$$-6 + 7d = 8 \implies 7d = 14 \implies d = 2.$$

Answer: $u_1 = -6, d = 2$

24. IB ref: 21M.2.SL.TZ2.3

The k th term is

$$u_k = 60 - 2.5(k - 1).$$

Set this equal to zero:

$$60 - 2.5(k - 1) = 0 \implies k - 1 = 24 \implies k = 25.$$

The terms are positive through $u_{24} = 2.5$, and $u_{25} = 0$. Hence $S_{24} = S_{25}$ is the maximum.

$$S_{24} = \frac{24}{2}(60 + 2.5) = 12(62.5) = 750.$$

Answer: $k = 25$; maximum $S_n = 750$

25. IB ref: 20N.1.SL.TZ0.T_15

Let the first slice volume be a and ratio be r .

$$u_2 = ar = 30, \quad u_5 = ar^4 = 240.$$

Dividing,

$$r^3 = 8 \implies r = 2.$$

Then

$$a = \frac{30}{2} = 15.$$

The whole pie gives

$$61425 = S_n = 15 \frac{2^n - 1}{2 - 1} = 15(2^n - 1).$$

Hence

$$2^n - 1 = 4095 \implies 2^n = 4096 = 2^{12}.$$

Thus $n = 12$.

Answer: $r = 2$, smallest slice 15 cm^3 , 12 slices

26. IB ref: 22N.2.SL.TZ0.4

The fourth term satisfies

$$50r^3 = 86.4,$$

so

$$r^3 = 1.728 = (1.2)^3 \implies r = 1.2.$$

Thus

$$S_n = 50 \frac{1.2^n - 1}{1.2 - 1} = 250(1.2^n - 1).$$

We require

$$250(1.2^n - 1) > 33500,$$

so

$$1.2^n > 135.$$

Therefore

$$n > \frac{\ln 135}{\ln 1.2} \approx 26.9045.$$

The smallest positive integer is

$$\boxed{27}.$$

27. IB ref: 20N.2.SL.TZ0.S_6

(a) The common ratio is

$$r = \frac{u_2}{u_1} = \frac{\frac{1}{4}a^2 - 3a}{a} = \frac{a}{4} - 3.$$

(b) A sum to infinity exists when $|r| < 1$:

$$-1 < \frac{a}{4} - 3 < 1.$$

Add 3:

$$2 < \frac{a}{4} < 4.$$

Therefore

$$\boxed{8 < a < 16}.$$

(c)

$$S_\infty = \frac{a}{1-r} = \frac{a}{1-(a/4-3)} = \frac{4a}{16-a}.$$

Set this equal to 76:

$$\frac{4a}{16-a} = 76.$$

Hence

$$4a = 1216 - 76a \implies 80a = 1216 \implies \boxed{a = \frac{76}{5}}.$$

This lies in $(8, 16)$, so it is admissible.

28. IB ref: 24M.1.AHL.TZ1.5

With first term 1 and ratio 10,

$$S_n = \frac{10^n - 1}{10 - 1} = \boxed{\frac{10^n - 1}{9}}.$$

Now

$$S_1 + S_2 + \cdots + S_n = \sum_{k=1}^n \frac{10^k - 1}{9}.$$

Thus

$$= \frac{1}{9} \left(\sum_{k=1}^n 10^k - n \right).$$

Since

$$\sum_{k=1}^n 10^k = 10 \frac{10^n - 1}{9},$$

we obtain

$$S_1 + \cdots + S_n = \frac{1}{9} \left(\frac{10(10^n - 1)}{9} - n \right) = \boxed{\frac{10(10^n - 1) - 9n}{81}}.$$

29. IB ref: 23M.1.AHL.TZ1.10

We are given

$$S_n = n^2 + 4n.$$

(a)

$$S_5 = 25 + 20 = 45.$$

Since $S_6 = 60$,

$$u_6 = S_6 - S_5 = 60 - 45 = 15.$$

(b)

$$u_1 = S_1 = 1 + 4 = 5.$$

(c) For $n \geq 2$,

$$u_n = S_n - S_{n-1}.$$

Now

$$S_{n-1} = (n-1)^2 + 4(n-1) = n^2 + 2n - 3.$$

Hence

$$u_n = (n^2 + 4n) - (n^2 + 2n - 3) = \boxed{2n + 3}.$$

This also gives $u_1 = 5$, so the formula holds for all positive integers n .

For the GP (v_n) ,

$$v_2 = u_1 = 5, \quad v_4 = u_6 = 15.$$

Thus

$$\frac{v_4}{v_2} = r^2 = 3,$$

so

$$r = \pm\sqrt{3}.$$

Since

$$v_{99} = v_2 r^{97},$$

and 97 is odd, the condition $v_{99} < 0$ forces

$$r = -\sqrt{3}.$$

29. conclusion and 30. IB ref: 24M.1.AHL.TZ2.10

For Question 29, the sign condition forces the negative ratio, so

$$v_5 = v_2 r^3 = 5(-\sqrt{3})^3 = -15\sqrt{3}.$$

Answer to Q29: $S_5 = 45$, $u_6 = 15$, $u_1 = 5$, $u_n = 2n + 3$, $r = \pm\sqrt{3}$, $v_5 = -15\sqrt{3}$.

30(a) The arithmetic sequence (u_n) begins

$$u_1, p, q, \dots$$

Consecutive arithmetic terms have the same difference, so

$$p - u_1 = q - p \implies \boxed{2p - q = u_1}.$$

(b) The geometric sequence (v_n) begins

$$v_1, s, t, \dots$$

Equal consecutive ratios give

$$\frac{s}{v_1} = \frac{t}{s} \implies \boxed{s^2 = v_1 t}.$$

(c) The two sequences have the same first term, so $u_1 = v_1$, and $q = t = 1$. From part (b),

$$s^2 = v_1 > 0.$$

Hence $u_1 > 0$. From part (a),

$$2p - 1 = u_1 > 0,$$

therefore

$$\boxed{p > \frac{1}{2}}.$$

(d) Take $u_1 = v_1 = 9$ and $q = t = 1$. For the AP, the first and third terms are 9 and 1, so the middle term is their arithmetic mean:

$$p = \frac{9 + 1}{2} = 5.$$

Thus $d = -4$ and the first four AP terms are

$$\boxed{9, 5, 1, -3}.$$

For the GP,

$$s^2 = v_1 t = 9.$$

Since $s > 0$, $s = 3$, so the common ratio is $1/3$ and the first four GP terms are

$$\boxed{9, 3, 1, \frac{1}{3}}.$$

(e) The new arithmetic sequence is (w_n) , with

$$w_1 = 9 + \ln 9 = 9 + 2 \ln 3, \quad w_2 = 5 + \ln 3, \quad w_3 = 1.$$

Hence

$$d_w = w_2 - w_1 = \boxed{-4 - \ln 3}.$$

30. continued

(f) We now sum the first ten terms of (w_n) . Using

$$S_{10} = \frac{10}{2} [2w_1 + 9d_w],$$

we obtain

$$\begin{aligned}\sum_{i=1}^{10} w_i &= 5 [2(9 + 2 \ln 3) + 9(-4 - \ln 3)] \\ &= 5(18 + 4 \ln 3 - 36 - 9 \ln 3) \\ &= 5(-18 - 5 \ln 3).\end{aligned}$$

Therefore

$$\boxed{\sum_{i=1}^{10} w_i = -90 - 25 \ln 3.}$$

Notation check. The AP and GP in the first part are different sequences, so writing their first terms as u_1 and v_1 keeps the roles clear. The logarithmic sequence is a third sequence, so it is labelled (w_n) . This avoids reusing one letter for several unrelated objects.

Continue to Section E - Additional IB Higher Level Past-Paper Solutions.

Section E - Additional IB Higher Level Past-Paper Solutions

HL MATERIAL. Full solutions to Questions 31–35. The worksheet questions are paraphrased/adapted from the cited IB Mathematics AA HL papers.

31. IB ref. 22M.1.AHL.TZ2.1

The rule is $u_n = 15 - 3n$.

(a) $u_1 = 15 - 3(1) = \boxed{12}$.

(b) Set $15 - 3n = -33$:

$$-3n = -48 \quad \Rightarrow \quad n = \boxed{16}.$$

(c)

$$d = u_{n+1} - u_n = [15 - 3(n+1)] - (15 - 3n) = \boxed{-3}.$$

32. IB ref. 25M.1.AHL.TZ1.5

Each length is multiplied by $\frac{3}{2}$ from one frame to the next, so area is multiplied by

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

Since $A_1 = 4 \cdot 5 = 20$,

$$A_n = 20 \left(\frac{9}{4}\right)^{n-1}.$$

For ten frames,

$$S_{10} = 20 \frac{(9/4)^{10} - 1}{(9/4) - 1} = 16 \left[\left(\frac{9}{4}\right)^{10} - 1 \right].$$

Hence the mean area is

$$\frac{8}{5} \left[\left(\frac{9}{4}\right)^{10} - 1 \right] \text{ cm}^2.$$

With ten ordered areas, the median is the average of A_5 and A_6 :

$$\begin{aligned} \text{median} &= \frac{1}{2} \left[20 \left(\frac{9}{4}\right)^4 + 20 \left(\frac{9}{4}\right)^5 \right] \\ &= 10 \left(\frac{9}{4}\right)^4 \left(1 + \frac{9}{4}\right) = \frac{65}{2} \left(\frac{9}{4}\right)^4 \text{ cm}^2. \end{aligned}$$

33. IB ref. 25M.2.AHL.TZ2.7 - GDC

Let the geometric ratio be r . Then

$$80r^3 = 0.74088 \Rightarrow r^3 = 0.009261 \Rightarrow r = 0.21.$$

Therefore

$$u_2 = 80(0.21) = 16.8.$$

For the arithmetic sequence, $u_1 = 80$ and $u_{11} = 16.8$, so

$$80 + 10d = 16.8 \Rightarrow d = -6.32.$$

Thus

$$S_n = \frac{n}{2} [160 - 6.32(n - 1)] = -3.16n^2 + 83.16n.$$

The continuous quadratic has its vertex at

$$n = \frac{83.16}{6.32} \approx 13.158.$$

Since n must be a positive integer, compare the neighbouring values:

$$S_{13} = 547.04, \quad S_{14} = 544.88.$$

Hence the greatest possible partial sum is

$$547.04.$$

IB point: a GDC can locate the vertex, but the sequence index is discrete; the adjacent integer values must be checked.

34. IB ref. 25M.1.AHL.TZ3.10

(a) Arithmetic means equal consecutive differences:

$$(3 - 2k) - (k - 5) = (5k + 3) - (3 - 2k).$$

So $8 - 3k = 7k$, hence

$$\boxed{k = \frac{4}{5}}, \quad u_3 = 5 \left(\frac{4}{5} \right) + 3 = \boxed{7}.$$

(b) For $k = 12$,

$$u_1 = 7, \quad u_2 = -21, \quad u_3 = 63.$$

The consecutive ratios are both -3 , so the terms begin a GP with $r = -3$. Since $|r| = 3 > 1$, the terms do not tend to 0 and the infinite series diverges.

(c) For a GP, the middle term squared equals the product of its neighbours:

$$(3 - 2k)^2 = (k - 5)(5k + 3).$$

Expanding gives

$$9 - 12k + 4k^2 = 5k^2 - 22k - 15 \quad \Rightarrow \quad \boxed{k^2 - 10k - 24 = 0}.$$

Thus $(k - 12)(k + 2) = 0$. The second value is $k = -2$, giving

$$\boxed{-7, 7, -7},$$

with common ratio -1 . Every pair sums to zero, hence

$$\boxed{S_{2m} = 0}.$$

35. IB ref. 22M.1.AHL.TZ1.10

(a) If the series is geometric, equality of consecutive ratios gives

$$\frac{p \ln x}{\ln x} = \frac{(1/3) \ln x}{p \ln x} \Rightarrow p^2 = \frac{1}{3}.$$

Thus

$$p = \pm \frac{1}{\sqrt{3}}.$$

In either case $|p| = 1/\sqrt{3} < 1$, so the geometric series converges.When $p > 0$, $r = 1/\sqrt{3}$ and

$$3 + \sqrt{3} = \frac{\ln x}{1 - 1/\sqrt{3}}.$$

Since $(3 + \sqrt{3})(1 - 1/\sqrt{3}) = 2$,

$$\ln x = 2 \Rightarrow \boxed{x = e^2}.$$

(b) For an arithmetic sequence, the middle of three consecutive terms is their arithmetic mean:

$$2p \ln x = \ln x + \frac{1}{3} \ln x.$$

Since $\ln x \neq 0$,

$$p = \frac{2}{3}.$$

Hence

$$d = \frac{2}{3} \ln x - \ln x = \boxed{-\frac{1}{3} \ln x}.$$

Now

$$S_n = \frac{n}{2} \left[2 \ln x + (n-1) \left(-\frac{1}{3} \ln x \right) \right].$$

Given $S_n = \ln(x^{-3}) = -3 \ln x$, divide by $\ln x > 0$:

$$\frac{n}{2} \left(2 - \frac{n-1}{3} \right) = -3.$$

Therefore

$$\frac{n(7-n)}{6} = -3 \Rightarrow n^2 - 7n - 18 = 0 \Rightarrow (n-9)(n+2) = 0.$$

The index is a positive integer, so

$$\boxed{n = 9}.$$

End of worked solutions.

36. CF Maths original synthesis - full solution

(a) The arithmetic sequence (u_n) has first term u_1 and common difference d , so

$$u_k = u_1 + (k - 1)d.$$

Therefore

$$u_{2n-1} = u_1 + (2n - 2)d, \quad u_{2n} = u_1 + (2n - 1)d.$$

Adding gives

$$x_n = u_{2n-1} + u_{2n} = 2u_1 + (4n - 3)d.$$

This is linear in n , and more directly

$$x_{n+1} - x_n = 4d,$$

so (x_n) is arithmetic with common difference $4d$.

(b) Use the information about the new sequence:

$$x_1 = 2u_1 + d = 11, \quad x_4 = 2u_1 + 13d = 47.$$

Subtracting,

$$12d = 36 \implies d = 3.$$

Then $2u_1 + 3 = 11$, so $u_1 = 4$. Hence

$$\boxed{x_n = 12n - 1}.$$

(c) Let the positive GP be

$$v_n = v_1 r^{n-1}.$$

Then

$$\begin{aligned} y_n &= v_{2n-1} + v_{2n} \\ &= v_1 r^{2n-2} + v_1 r^{2n-1} \\ &= v_1 (1 + r) r^{2n-2}. \end{aligned}$$

Thus

$$\frac{y_{n+1}}{y_n} = r^2,$$

so (y_n) is geometric with common ratio r^2 .

(d) Since

$$\frac{y_2}{y_1} = \frac{30}{120} = \frac{1}{4},$$

we have $r^2 = 1/4$. Because $0 < r < 1$, $r = 1/2$. Also

$$y_1 = v_1(1 + r) = \frac{3}{2}v_1 = 120,$$

so $v_1 = 80$. Therefore

$$\boxed{y_n = 120 \left(\frac{1}{4}\right)^{n-1}}.$$

(e)

$$z_1 = 131, \quad z_2 = 53, \quad z_3 = \frac{85}{2}, \quad z_4 = \frac{391}{8}.$$

The values decrease and then increase, suggesting a minimum near $n = 3$.

36. continued

(f) Since $x_{n+1} - x_n = 12$ and $y_{n+1} = y_n/4$,

$$z_{n+1} - z_n = 12 + \frac{120}{4^n} - \frac{120}{4^{n-1}} = 12 - \frac{90}{4^{n-1}}.$$

For $n = 1, 2$ this is negative, while for $n = 3$ it is positive:

$$12 - 90 < 0, \quad 12 - \frac{90}{4} < 0, \quad 12 - \frac{90}{16} > 0.$$

After $n = 3$, the subtracted fraction only gets smaller, so all later differences are positive. Hence the unique minimum occurs at

$$\boxed{n = 3}.$$

(g)

$$t_n = 12 - (z_{n+1} - z_n) = \frac{90}{4^{n-1}}.$$

Thus (t_n) is geometric with $t_1 = 90$ and ratio $1/4$. Since $|1/4| < 1$,

$$\boxed{\sum_{n=1}^{\infty} t_n = \frac{90}{1 - 1/4} = 120}.$$

(h) Since $x_n = 12n - 1$,

$$X_N = \sum_{n=1}^N (12n - 1) = 6N^2 + 5N.$$

Also

$$Y_N = 120 \frac{1 - (1/4)^N}{1 - 1/4} = 160(1 - 4^{-N}).$$

Therefore

$$\boxed{Z_N = 6N^2 + 5N + 160(1 - 4^{-N})}.$$

(i) A GDC/table gives

$$Z_{27} \approx 4669 < 5000, \quad Z_{28} \approx 5004 > 5000.$$

So the least possible index is

$$\boxed{N = 28}.$$

(j) Since $Z_N = X_N + Y_N$,

$$Z_N - X_N = Y_N = 160(1 - 4^{-N}).$$

As $N \rightarrow \infty$, $4^{-N} \rightarrow 0$, so

$$\boxed{\lim_{N \rightarrow \infty} (Z_N - X_N) = 160}.$$

Big picture. The notation u_1 and v_1 keeps the two starting sequences separate. Pairing terms creates an AP from one sequence and a GP from the other; first differences then reveal another hidden GP.

37. Arithmetic sums form a geometric sequence

Let the AP have first term u_1 and fourth term u_4 .

(a) Since

$$u_4 = u_1 + 3d,$$

$$d = \frac{u_4 - u_1}{3}.$$

(b) Use

$$S_n = \frac{n}{2} [2u_1 + (n-1)d]$$

and substitute the same expression for d each time:

$$S_5 = \frac{5}{2} \left[2u_1 + 4 \left(\frac{u_4 - u_1}{3} \right) \right] = \frac{5}{3} (u_1 + 2u_4),$$

$$S_6 = 3 \left[2u_1 + 5 \left(\frac{u_4 - u_1}{3} \right) \right] = u_1 + 5u_4,$$

$$S_9 = \frac{9}{2} \left[2u_1 + 8 \left(\frac{u_4 - u_1}{3} \right) \right] = 3(4u_4 - u_1).$$

(c) If S_5, S_6, S_9 are consecutive GP terms, the middle term squared equals the product of the outer terms:

$$S_6^2 = S_5 S_9.$$

Thus

$$(u_1 + 5u_4)^2 = \frac{5}{3} (u_1 + 2u_4) 3(4u_4 - u_1) = 5(u_1 + 2u_4)(4u_4 - u_1).$$

Expanding,

$$u_1^2 + 10u_1u_4 + 25u_4^2 = -5u_1^2 + 10u_1u_4 + 40u_4^2.$$

The mixed terms cancel, giving

$$6u_1^2 = 15u_4^2 \implies \boxed{2u_1^2 = 5u_4^2}.$$

(d) If $u_1, u_4 > 0$, take the positive square root:

$$\boxed{\frac{u_4}{u_1} = \sqrt{\frac{2}{5}}}.$$

Since this is less than 1, the fourth term is smaller than the first, so the AP is decreasing.

Why the method works. First do all the arithmetic-sequence algebra. Only after the sums are written in the same two variables do we impose the geometric condition $\text{middle}^2 = \text{first} \times \text{third}$.

38. Exam-style bridge: first and third AP terms

The first and third terms are u_1 and u_3 . Hence

$$u_3 = u_1 + 2d \implies d = \frac{u_3 - u_1}{2}.$$

(a) The fourth term is

$$u_4 = u_1 + 3d = u_1 + \frac{3}{2}(u_3 - u_1) = \frac{3u_3 - u_1}{2}.$$

Therefore

$$\begin{aligned} S_4 &= u_1 + \frac{u_1 + u_3}{2} + u_3 + \frac{3u_3 - u_1}{2} \\ &= \boxed{u_1 + 3u_3}. \end{aligned}$$

(b) We also need S_5 and S_7 . The fifth term is

$$u_5 = u_1 + 4d = 2u_3 - u_1,$$

so

$$S_5 = S_4 + u_5 = (u_1 + 3u_3) + (2u_3 - u_1) = 5u_3.$$

Also

$$S_7 = \frac{7}{2}[2u_1 + 6d] = \frac{7}{2}(3u_3 - u_1).$$

Because S_4, S_5, S_7 are consecutive GP terms,

$$S_5^2 = S_4 S_7.$$

Thus

$$25u_3^2 = (u_1 + 3u_3)\frac{7}{2}(3u_3 - u_1) = \frac{7}{2}(9u_3^2 - u_1^2).$$

Multiplying by 2,

$$50u_3^2 = 63u_3^2 - 7u_1^2,$$

so

$$\boxed{7u_1^2 = 13u_3^2}.$$

(c) For $u_1, u_3 > 0$,

$$\boxed{\frac{u_3}{u_1} = \sqrt{\frac{7}{13}}}.$$

This is less than 1, so $u_3 < u_1$ and the arithmetic sequence is decreasing.

Exam habit. The letters u_1 and u_3 tell you positions, not just values. Recover d from the gap of two indices before building the partial sums.

39. A formula for S_n hides both an AP and a GP

We are given

$$S_n = n^2 + pn.$$

(a) For $n \geq 2$,

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= (n^2 + pn) - [(n-1)^2 + p(n-1)] \\ &= 2n + p - 1. \end{aligned}$$

This also agrees with $u_1 = S_1 = p + 1$, so

$$\boxed{u_n = 2n + p - 1}.$$

(b)

$$u_{n+1} - u_n = 2,$$

so (u_n) is arithmetic with

$$\boxed{d = 2}.$$

(c)

$$u_2 = p + 3, \quad u_5 = p + 9, \quad u_9 = p + 17.$$

These are consecutive GP terms, so

$$(p + 9)^2 = (p + 3)(p + 17).$$

Expanding,

$$p^2 + 18p + 81 = p^2 + 20p + 51,$$

therefore

$$\boxed{p = 15}.$$

(d) The three GP terms are

$$18, 24, 32,$$

so

$$\boxed{r = \frac{24}{18} = \frac{4}{3}}.$$

(e) Instead of adding eleven terms one by one, use partial sums:

$$u_{10} + u_{11} + \cdots + u_{20} = S_{20} - S_9.$$

With $p = 15$,

$$S_{20} = 20^2 + 15(20) = 700, \quad S_9 = 9^2 + 15(9) = 216.$$

Hence

$$\boxed{u_{10} + u_{11} + \cdots + u_{20} = 484}.$$

Hidden structure. The partial-sum formula first creates the term formula; the term formula reveals an AP; selected AP terms then create a GP. One question can legitimately change sequence type several times.

40. Linking an AP to a convergent GP

The arithmetic sequence satisfies $u_2 = 18$ and $S_5 = 70$.

(a) Since $u_2 = u_1 + d$,

$$u_1 + d = 18. \quad (1)$$

Also

$$S_5 = \frac{5}{2}[2u_1 + 4d] = 5(u_1 + 2d) = 70,$$

so

$$u_1 + 2d = 14. \quad (2)$$

Subtracting (1) from (2) gives $d = -4$, hence

$$\boxed{u_1 = 22, \quad d = -4}.$$

(b)

$$u_5 = 22 + 4(-4) = \boxed{6}.$$

(c) The positive GP has

$$v_1 = u_2 = 18, \quad v_3 = u_5 = 6.$$

Thus

$$18r^2 = 6 \implies r^2 = \frac{1}{3}.$$

Positivity gives

$$\boxed{r = \frac{1}{\sqrt{3}}}.$$

(d) The crucial check comes before the formula:

$$\left| \frac{1}{\sqrt{3}} \right| < 1.$$

Therefore the terms shrink in magnitude and the infinite GP converges. Hence

$$\begin{aligned} S_\infty &= \frac{18}{1 - 1/\sqrt{3}} \\ &= \frac{18\sqrt{3}}{\sqrt{3} - 1} \\ &= \boxed{27 + 9\sqrt{3}}. \end{aligned}$$

(e) For a convergent GP,

$$\frac{S_n}{S_\infty} = 1 - r^n.$$

We need $S_n > 0.95S_\infty$, so

$$1 - r^n > 0.95 \iff r^n < 0.05.$$

Now

$$r^5 = \frac{1}{9\sqrt{3}} \approx 0.0642 > 0.05, \quad r^6 = \frac{1}{27} \approx 0.0370 < 0.05.$$

Therefore the least possible integer is

$$\boxed{n = 6}.$$

Why infinity is finite here. Each new GP term is only about 57.7% of the previous one. The extra pieces never stop, but they become so small that the running total approaches a fixed ceiling.

41. Two different middle-term rules

The AP has $u_2 = 14$ and $u_6 = 2$.

(a) The index gap is 4, so

$$4d = u_6 - u_2 = 2 - 14 = -12 \implies d = -3.$$

Then

$$u_1 = u_2 - d = 14 - (-3) = \boxed{17}.$$

(b)

$$S_3 = \frac{3}{2}[2(17) + 2(-3)] = 42,$$

$$S_4 = 2[2(17) + 3(-3)] = 50,$$

$$S_6 = 3[2(17) + 5(-3)] = 57.$$

Hence

$$\boxed{S_3 = 42, \quad S_4 = 50, \quad S_6 = 57}.$$

(c) If 42, 50, 57 were consecutive AP terms, the middle term would satisfy

$$2(50) = 42 + 57.$$

But $100 \neq 99$, so they are **not** consecutive terms of an AP.

(d) If they were consecutive GP terms, the middle term would satisfy

$$50^2 = 42 \cdot 57.$$

But

$$2500 \neq 2394,$$

so they are **not** consecutive terms of a GP.

(e)

$$u_5 = 17 + 4(-3) = 5, \quad u_6 = 2.$$

Thus the new GP has

$$v_1 = |u_5| = 5, \quad v_2 = |u_6| = 2,$$

so

$$\boxed{r = \frac{2}{5}}.$$

Since $|r| = 2/5 < 1$, the sum to infinity exists.

(f)

$$S_\infty = \frac{5}{1 - 2/5} = \boxed{\frac{25}{3}}.$$

The series may contain infinitely many terms because each new term is only $2/5$ of the previous one; the terms shrink to zero quickly enough that the partial sums approach $25/3$ instead of growing without bound.

Two middle rules to remember. For consecutive AP terms a, b, c , use $2b = a + c$. For consecutive GP terms, use $b^2 = ac$. They look similar, but one comes from equal differences and the other from equal ratios.