

Sequences Homework Worksheet

Sequences, Arithmetic and Geometric Sequences

Questions only • Increasing difficulty • Full working expected

Instructions. Attempt the questions in order. Show algebraic working clearly, justify “show that” statements, and state all admissible solutions where a sign choice is possible. Questions marked **GDC** are intended for calculator use.

Source note.

Questions 1-22 are teacher-selected textbook practice. Questions 23-35 are paraphrased/adapted IB past-paper practice, with the original references shown for lookup. Section F (Questions 36-41) contains original CF Maths combined AP/GP synthesis. Notation is standardised so the first term is written as u_1 (or v_1 for a second sequence).

Section A - Foundations and Core Technique

The questions begin with notation, direct formula use, and standard arithmetic/geometric sequence calculations.

1. Consider the number sequence

$$5, 9, 11, 12, 15, 19, \dots$$

Find (a) u_2 , (b) u_6 , (c) S_4 .

2. Find k if $3k$, $k - 2$, and $k + 7$ are consecutive terms of an arithmetic sequence.

3. The sequence is defined by

$$u_n = 6 \left(\frac{1}{2} \right)^{n-1}.$$

- (a) Prove that the sequence is geometric.
(b) Find u_1 and the common ratio r .
(c) **GDC** Find the 16th term to three significant figures.

4. A geometric sequence has sixth term $\frac{16}{3}$ and tenth term $\frac{256}{3}$. Determine all possible real general terms u_n .

5. Find the sum of:

- (a) the first 50 positive multiples of 11;
(b) the multiples of 7 between 0 and 1000;
(c) the integers from 1 to 100 which are not divisible by 3;
(d) the three-digit positive integers which start or end with the digit 4.

6. The expressions $k - 1$, $2k + 3$, and $27 - k$ are the first three terms of an arithmetic sequence.

- (a) Find k .
- (b) Find the sum of the first 25 terms of the sequence.
7. The sixth term of an arithmetic sequence is 21, and the sum of the first seventeen terms is 0. Find the first two terms of the sequence.
8. An arithmetic series satisfies $S_3 = 9$ and $S_6 = 90$. Find S_{10} .

Section B - Structure, Series and Connections

These problems require you to recognise hidden arithmetic/geometric structure and combine several facts before calculating.

9. Suppose $u_1, u_2, u_3, u_4, u_5, u_6, \dots$ is an arithmetic sequence.

(a) Show that

$$u_1 + u_2, \quad u_3 + u_4, \quad u_5 + u_6, \dots$$

is also an arithmetic sequence.

(b) Given that $u_2 = 5$ and $u_{10} = 61$, find the 30th term of the sequence in part (a).

10. The numbers $\frac{1}{k}$, k , and $k^2 + 1$ are respectively the 3rd, 4th, and 6th terms of an arithmetic sequence. Given that $k \in \mathbb{Q}$, find:

(a) k ;

(b) the general term u_n .

11. A geometric series has second term 6. The sum of its first three terms is -14 . Find all possible values of its fourth term.

12. An arithmetic sequence and a geometric sequence both have first term 1, and their second terms are equal. The 14th term of the arithmetic sequence is three times the third term of the geometric sequence. Find the 20th term of each sequence, giving all possible solutions.

13. An infinite geometric series has

$$S_{\infty} = \frac{64}{3}, \quad S_3 = 21.$$

Find S_5 .

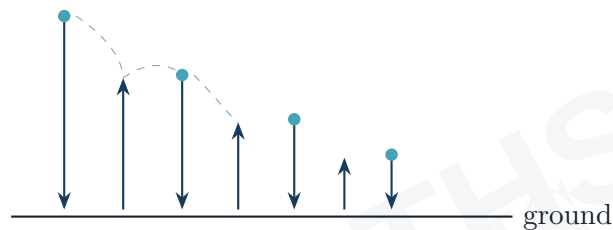
14. When dropped, a ball takes 1 second to hit the ground. It then takes 90% of this time to rebound to its new height, and the same time to fall back to the ground. This pattern continues indefinitely.

(a) Show that the total time of motion is represented by

$$1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \dots$$

(b) Let S_n denote the sum of the first n displayed terms. Find a formula for S_n .

(c) How long does it take for the ball to come to rest?



15. Let S_n denote the sum of the first n terms of an arithmetic sequence. The three numbers S_1 , S_2 , and S_4 form a geometric sequence. Find all possible common ratios of this geometric sequence.

16. The numbers x, y, z are consecutive terms of a geometric sequence and satisfy

$$x + y + z = \frac{7}{3}, \quad x^2 + y^2 + z^2 = \frac{91}{9}.$$

Find all possible ordered triples (x, y, z) .

17. An arithmetic sequence has common difference d . The partial sums S_2 , S_5 , and S_7 themselves form an arithmetic sequence. Find, in terms of d , the common difference of the sequence S_2, S_5, S_7 .

Section C - Proof, Sigma Notation and Extension

The final teacher-selected questions emphasise proof, interpretation, and connections between sequences and other algebraic ideas.

18. Let $u_n = 3 + 2n$.

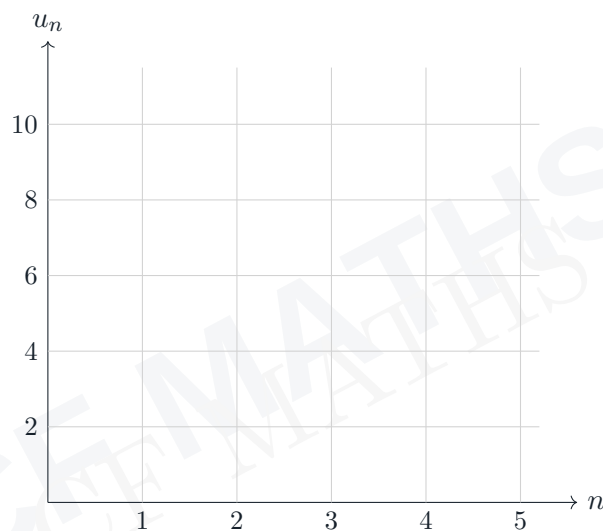
(a) For $n = 1, \dots, 4$, plot the points (n, u_n) and draw rectangles with vertices

$$(n, u_n), (n+1, u_n), (n, 0), (n+1, 0).$$

(b) Explain how S_n relates to the areas of the rectangles.

(c) Using your sketch, explain why

$$(i) \quad u_{n+1} = u_n + 2, \quad (ii) \quad S_{n+1} = S_n + u_{n+1}.$$



19.

(a) Show that

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1.$$

(b) Write the identity successively for $x = 0, 1, 2, 3, \dots, n$.

(c) Add the resulting equations vertically and hence show that

$$(n+1)^3 = 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + (n+1).$$

(d) Given that

$$\sum_{k=1}^n k = \frac{n(n+1)}{2},$$

show that

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

20. Given that

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6},$$

write

$$\sum_{k=1}^n (k+1)(k+2)$$

in simplest form. Check your formula in the case $n = 10$.

21. The three positive numbers x, y, z satisfy $x > y > z > 0$. Show that if

$$\frac{1}{x}, \quad \frac{1}{y}, \quad \frac{1}{z}$$

are consecutive terms of an arithmetic sequence, then the three numbers

$$x - z, \quad y, \quad x - y + z$$

could be the side lengths of a right-angled triangle.

22. Show that

$$\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \cdots = 1.$$

Section D - IB Past-Paper Practice (8 questions)

The questions below are **paraphrased/adapted** from the cited IB Mathematics: Analysis and Approaches past-paper questions. The mathematical data and structure are retained, but the wording is not reproduced verbatim. Difficulty rises through the section.

23. IB ref: 21M.1.SL.TZ1.3 [5 marks]

For an arithmetic sequence, $u_8 = 8$ and $S_8 = 8$. Determine u_1 and the common difference d .

24. IB ref: 21M.2.SL.TZ2.3 [5 marks]**GDC**

An arithmetic sequence has first term 60 and common difference -2.5 .

- (a) The k th term is zero. Find k .
- (b) If S_n is the sum of the first n terms, determine the maximum value of S_n .

25. IB ref: 20N.1.SL.TZ0.T_15 [6 marks]

The volumes of consecutive slices cut from a pie form a geometric sequence, smallest slice first. The second slice has volume 30 cm^3 , the fifth has volume 240 cm^3 , and the total volume is 61425 cm^3 .

- (a) Find the common ratio.
- (b) Find the volume of the smallest slice.
- (c) Find the number of slices.

26. IB ref: 22N.2.SL.TZ0.4 [5 marks]**GDC**

A geometric sequence has first term 50 and fourth term 86.4. Let S_n be the sum of the first n terms. Find the smallest positive integer n for which $S_n > 33500$.

27. IB ref: 20N.2.SL.TZ0.S_6 [8 marks]**GDC**

An infinite geometric series has first term u_1 and second term

$$u_2 = \frac{1}{4}u_1^2 - 3u_1, \quad u_1 > 0.$$

- (a) Find the common ratio in terms of u_1 .
- (b) Find the values of u_1 for which the sum to infinity exists.
- (c) Find u_1 if $S_\infty = 76$.

28. IB ref: 24M.1.AHL.TZ1.5 [5 marks]

A geometric sequence has first term 1 and common ratio 10. Let S_n denote the sum of its first n terms.

- (a) Find an expression for S_n .
- (b) Hence show that

$$S_1 + S_2 + \cdots + S_n = \frac{10(10^n - 1) - 9n}{81}.$$

29. IB ref: 23M.1.AHL.TZ1.10 [14 marks]

For an arithmetic sequence (u_n) , the partial sums satisfy

$$S_n = n^2 + 4n.$$

- (a) Find S_5 and then use $S_6 = 60$ to find u_6 .
- (b) Find u_1 .
- (c) Obtain a formula for u_n in terms of n .

A geometric sequence (v_n) satisfies $v_2 = u_1$ and $v_4 = u_6$.

- (d) Find all possible real common ratios r .
- (e) Given that $v_{99} < 0$, find v_5 .

30. IB ref: 24M.1.AHL.TZ2.10 [16 marks]

An arithmetic sequence (u_n) begins $u_1, p, q, \dots$, with $u_1, p, q \neq 0$.

- (a) Show that $2p - q = u_1$.

A geometric sequence (v_n) begins $v_1, s, t, \dots$, with $v_1, s, t \neq 0$.

- (b) Show that $s^2 = v_1 t$.

The two sequences have the same first term, so $u_1 = v_1$, and $q = t = 1$.

- (c) Show that $p > \frac{1}{2}$.

- (d) Now take $u_1 = v_1 = 9$ and $s > 0$. Write the first four terms of each sequence.

Define a new arithmetic sequence (w_n) by the first three terms

$$w_1 = 9 + \ln 9, \quad w_2 = 5 + \ln 3, \quad w_3 = 1 + \ln 1.$$

- (e) Find its common difference in terms of $\ln 3$.

- (f) Show that

$$\sum_{i=1}^{10} w_i = -90 - 25 \ln 3.$$

Continue to Section E - Additional IB Higher Level Past-Paper Practice.

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HL MATERIAL. Questions 31–35 are additional **IB Mathematics: Analysis and Approaches Higher Level** past-paper practice. The wording has been paraphrased/adapted for this CF Maths worksheet; the original paper reference is shown for lookup.

31. An arithmetic sequence has general term

$$u_n = 15 - 3n.$$

- (a) State the value of u_1 .
- (b) A term of the sequence is -33 . Find its position n .
- (c) Find the common difference d .

IB ref. 22M.1.AHL.TZ2.1 [5 marks]

Working space

HL MATERIAL. Higher Level past-paper practice.

32. Ten rectangular frames are labelled $F_1, F_2, \dots, F_{10}$. Frame F_1 has width 4 cm and height 5 cm. To obtain each successive frame, both dimensions are multiplied by 1.5.

Let A_n denote the area of F_n .

(a) (i) Show that

$$A_n = 20 \left(\frac{9}{4} \right)^{n-1} \text{ cm}^2.$$

(ii) Find the **mean** area of the ten frames, giving an exact answer.

(b) Find the **median** area of the ten frames, giving an exact answer.

IB ref. 25M.1.AHL.TZ1.5 [8 marks]

Working space

HL MATERIAL - GDC. A graphical/table method is appropriate in part (b); the final answer must respect the integer domain of the sequence index.

33. A geometric sequence has first term 80 and fourth term 0.74088.

- (a) Find the second term of the geometric sequence.
- (b) The first and second terms of this geometric sequence are also, respectively, the first and eleventh terms of an arithmetic sequence. Let S_n be the sum of the first n terms of this arithmetic sequence. Find the greatest possible value of S_n , correct to two decimal places.

IB ref. 25M.2.AHL.TZ2.7 [7 marks]

CF MATHS

Working space

HL MATERIAL. Higher Level past-paper practice.

34. A sequence $\{u_n\}$ has first three terms

$$u_1 = k - 5, \quad u_2 = 3 - 2k, \quad u_3 = 5k + 3,$$

where $k \in \mathbb{R}$.

(a) Suppose $\{u_n\}$ is arithmetic.

(i) Find k .

(ii) Hence find u_3 .

(b) Take $k = 12$.

(i) Show that the first three terms form a geometric sequence.

(ii) Explain why the corresponding infinite geometric series has no sum to infinity.

(c) The sequence is geometric for one further value of k .

(i) Show that this value satisfies $k^2 - 10k - 24 = 0$.

(ii) Find the first three terms for this second geometric case.

(iii) Hence write down S_{2m} , the sum of the first $2m$ terms.

IB ref. 25M.1.AHL.TZ3.10 [16 marks]

HL MATERIAL. Higher Level past-paper practice.

35. Let $x > 1$ and $p \neq 0$. Consider the series

$$\ln x + p \ln x + \frac{1}{3} \ln x + \cdots .$$

(a) First suppose the series is geometric.

- (i) Show that $p = \pm \frac{1}{\sqrt{3}}$.
- (ii) Hence explain why the geometric series converges.
- (iii) Given $p > 0$ and $S_{\infty} = 3 + \sqrt{3}$, find x .

(b) Now suppose the same first three terms belong to an arithmetic series with common difference d .

- (i) Show that $p = \frac{2}{3}$.
- (ii) Express d in the form $k \ln x$, where $k \in \mathbb{Q}$.
- (iii) The sum of the first n terms is $\ln\left(\frac{1}{x^3}\right)$. Find n .

IB ref. 22M.1.AHL.TZ1.10 [18 marks]

Section F - Combined AP/GP Synthesis

These questions deliberately mix arithmetic sequences, geometric sequences, partial sums and new sequences built from old ones. Keep the notation attached to the correct sequence: u_1 is the first term of (u_n) , v_1 is the first term of (v_n) .

Notation convention. Throughout this section, the first term is written explicitly as u_1 (or v_1 for a second sequence), rather than introducing a separate letter such as a .

36. CF Maths original synthesis [28 marks]

An arithmetic sequence (u_n) has first term u_1 and common difference $d > 0$. Define

$$x_n = u_{2n-1} + u_{2n}, \quad n \geq 1,$$

and suppose $x_1 = 11$ and $x_4 = 47$.

- (a) Show that (x_n) is arithmetic, and express its common difference in terms of d . [3]
- (b) Hence find u_1 and d , and write x_n explicitly in terms of n . [3]

A positive geometric sequence (v_n) has first term v_1 and common ratio r , where $0 < r < 1$. Define

$$y_n = v_{2n-1} + v_{2n},$$

and suppose $y_1 = 120$ and $y_2 = 30$.

- (c) Show that (y_n) is geometric and state its common ratio in terms of r . [2]
- (d) Find r and v_1 , and hence show that

$$y_n = 120 \left(\frac{1}{4} \right)^{n-1}.$$

Now define $z_n = x_n + y_n$.

- (e) Find the exact values of z_1, z_2, z_3, z_4 . What do they suggest about the behaviour of (z_n) ? [2]
- (f) Show that

$$z_{n+1} - z_n = 12 - \frac{90}{4^{n-1}}.$$

Hence prove that (z_n) has a unique minimum, and state the index at which it occurs. [4]

36. continued

The difference between the constant increase 12 and the actual change in (z_n) defines

$$t_n = 12 - (z_{n+1} - z_n).$$

(g) Show that (t_n) is geometric. Find its first term, common ratio and sum to infinity. [3]

(h) Let

$$X_N = \sum_{n=1}^N x_n, \quad Y_N = \sum_{n=1}^N y_n, \quad Z_N = \sum_{n=1}^N z_n.$$

Find exact formulae for X_N, Y_N and Z_N . [4]

(i) **GDC** Find the least positive integer N for which $Z_N > 5000$. Justify your answer by checking the two neighbouring integer values. [2]

(j) Explain why $Z_N - X_N$ approaches a finite limit even though both Z_N and X_N grow without bound. Find that limit. [2]

Big idea. A complicated-looking sequence can become much simpler after you form a new sequence from its pairs, differences or partial sums. In combined questions, always ask: *what is the new object, and is it AP, GP, or neither?*

Combined Exercises - partial sums becoming geometric

The next questions are original CF Maths problems inspired by the kind of multi-stage reasoning used in IB examinations. They are not copies of official wording.

37. Arithmetic sums form a geometric sequence [11 marks]

An arithmetic sequence (u_n) has first term u_1 and fourth term u_4 . Let S_n be the sum of the first n terms.

- (a) Express the common difference d in terms of u_1 and u_4 . [2]
- (b) Find S_5, S_6 and S_9 in terms of u_1 and u_4 . [4]
- (c) Given that S_5, S_6, S_9 , in that order, are consecutive terms of a geometric sequence, show that

$$2u_1^2 = 5u_4^2.$$

[4]

- (d) If $u_1 > 0$ and $u_4 > 0$, find the exact value of $\frac{u_4}{u_1}$. [1]

38. Exam-style bridge: first and third AP terms [10 marks]

The first and third terms of an arithmetic sequence are u_1 and u_3 , respectively. The sum of the first n terms is S_n .

- (a) Find S_4 in terms of u_1 and u_3 . [3]
- (b) Given that S_4, S_5, S_7 are consecutive terms of a geometric sequence, show that

$$7u_1^2 = 13u_3^2.$$

[5]

- (c) If $u_1, u_3 > 0$, find u_3/u_1 exactly and state whether the arithmetic sequence is increasing or decreasing. [2]

38. continued - working space**39. A formula for S_n hides both an AP and a GP** [13 marks]

A sequence has partial sums

$$S_n = n^2 + pn,$$

where p is a real constant. The terms u_2, u_5, u_9 , in that order, are consecutive terms of a geometric sequence.

- (a) Use $u_n = S_n - S_{n-1}$ to show that

$$u_n = 2n + p - 1.$$

[3]

- (b) Hence show that (u_n) is arithmetic and state its common difference.

[2]

- (c) Find p .

[3]

- (d) Find the common ratio of the geometric sequence u_2, u_5, u_9 .

[2]

- (e) Find $u_{10} + u_{11} + \cdots + u_{20}$.

[3]

40. Linking an AP to a convergent GP [13 marks]

An arithmetic sequence (u_n) satisfies

$$u_2 = 18, \quad S_5 = 70.$$

A positive geometric sequence (v_n) is defined so that

$$v_1 = u_2, \quad v_3 = u_5.$$

- (a) Find u_1 and the common difference d of the arithmetic sequence. [3]
- (b) Find u_5 . [1]
- (c) Find the common ratio r of the geometric sequence. [3]
- (d) Explain, using the value of r , why the geometric series has a sum to infinity. Hence find S_∞ exactly. [4]
- (e) Find the least positive integer n for which the partial sum of the geometric series exceeds 95% of S_∞ . Justify the integer choice. [2]

What this question is testing. The AP gives you two values that become non-consecutive terms of a GP. Once the GP ratio is found, the question changes again: now the key decision is whether $|r| < 1$ before using S_∞ .

Final Combined Challenge

One last question pulls together consecutive-term identities, partial sums and convergence.

41. Two different “middle term” rules [14 marks]

An arithmetic sequence (u_n) has $u_2 = 14$ and $u_6 = 2$.

(a) Find u_1 and d . [2]

(b) Find S_3, S_4, S_6 . [3]

A second sequence is formed from the three numbers S_3, S_4, S_6 .

(c) Determine whether these three numbers can be consecutive terms of an arithmetic sequence. Justify your answer using the arithmetic-middle condition. [2]

(d) Determine whether they can be consecutive terms of a geometric sequence. Justify your answer using the geometric-middle condition. [2]

Now define a geometric sequence (v_n) by $v_1 = |u_5|$ and $v_2 = |u_6|$.

(e) Find its common ratio and decide whether S_∞ exists. [2]

(f) If it exists, find the exact sum to infinity and explain in one sentence why the terms can continue forever while the total remains finite. [3]

End of Worksheet

Arithmetic Sequences • Geometric Sequences • Partial Sums • Combined Synthesis