



TMUA PAPER 1

# ALGEBRA & FUNCTIONS

Teacher Notes • Expanded Worked-Example Edition

**SPECIFICATION NOTE** This edition is built around the currently published UAT-UK TMUA algebra-and-functions requirements, especially rational indices, surds, quadratics, simultaneous linear/quadratic equations, inequalities, polynomial division by linear and quadratic divisors, Factor/Remainder Theorems, equations versus identities, functions as mappings, common graphs, transformations, composition and intersection/root reasoning. The additional rational-inequality, recursive-function and mixed-synthesis problems are CF Maths admissions-test enrichment constructed from these ideas.

**DESIGN OF THIS EDITION** The notes begin immediately with indices. Worked-example labels do not carry difficulty tags. The examples themselves are deliberately demanding, but the solutions favour transparent syllabus-level methods over opaque shortcuts. Graphs and sketches are used whenever they clarify roots, domains, transformations or modulus inequalities. Floor/ceiling integration remains outside this chapter.

# 1. Indices and Power Rules

Rational indices are usually not difficult because of the laws themselves; the difficulty is recognising what to simplify, what substitution to make, and which domain or integer condition survives after simplification.

## Core laws and high-value habits

For suitable real values,

$$a^m a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn},$$

$$a^{1/n} = \sqrt[n]{a}, \quad a^{m/n} = \sqrt[n]{a^m}, \quad a^{-r} = \frac{1}{a^r}.$$

For integer-valued expressions, prime-factorise first. A positive rational number is an integer exactly when every prime has a non-negative integer exponent after complete simplification.

### WORKED EXAMPLE 1 Integer condition after total cancellation

For non-zero integers  $p, q$ , determine exactly when

$$E = \frac{36^{p-q} 3^q}{12^{2p-q} 6^p}$$

is an integer.

#### SOLUTION

Write every base in powers of 2 and 3:

$$36 = 2^2 3^2, \quad 12 = 2^2 3, \quad 6 = 2 \cdot 3.$$

Hence

$$E = 2^{2(p-q)-2(2p-q)-p} 3^{2(p-q)+q-(2p-q)-p} = 2^{-3p} 3^{-p} = 24^{-p}.$$

The variable  $q$  disappears. Since  $p \neq 0$ ,  $24^{-p}$  is an integer exactly when  $-p > 0$ .

$$\boxed{p < 0}$$

**TMUA SHORTCUT.** Simplify completely before imposing any condition on the original exponents. The final prime exponents, not the visible exponents in the question, control integrality.

### WORKED EXAMPLE 2 Two-prime region in the parameter plane

For non-zero integers  $m, n$ , find the condition for

$$E = \frac{18^{2m-n} 8^{m+n}}{12^m 27^{m-n}}$$

to be an integer.

#### SOLUTION

Using  $18 = 2 \cdot 3^2$ ,  $8 = 2^3$ ,  $12 = 2^2 \cdot 3$  and  $27 = 3^3$ ,

$$E = 2^{(2m-n)+3(m+n)-2m} 3^{2(2m-n)-m-3(m-n)}.$$

So

$$E = 2^{3m+2n} 3^n.$$

Because  $m, n$  are integers,  $E$  is an integer precisely when both prime exponents are non-negative:

$$\boxed{n \geq 0 \quad \text{and} \quad 3m + 2n \geq 0},$$

with the original restrictions  $m, n \neq 0$  retained.

### WORKED EXAMPLE 3 Reciprocal powers without solving the quadratic

Suppose  $x > 0$  and

$$x^{1/3} + x^{-1/3} = 4.$$

Find  $x + x^{-1}$  exactly.

#### SOLUTION

Set  $t = x^{1/3} > 0$ . Then  $t + t^{-1} = 4$  and the target is  $t^3 + t^{-3}$ . Use

$$(t + t^{-1})^3 = t^3 + t^{-3} + 3(t + t^{-1}).$$

Therefore

$$64 = t^3 + t^{-3} + 12,$$

so

$$\boxed{x + x^{-1} = 52}.$$

**KEY IDEA.** The direct quadratic for  $t$  creates surds and then asks you to cube them. Symmetric-power recurrences avoid that entirely.

### WORKED EXAMPLE 4 A hidden cubic in a cube-root substitution

Solve over the reals

$$x - 7x^{2/3} + 14x^{1/3} - 8 = 0.$$

#### SOLUTION

Let  $t = x^{1/3}$ , which is valid for every real  $x$ . Then

$$t^3 - 7t^2 + 14t - 8 = 0.$$

The cubic factorises as

$$(t - 1)(t - 2)(t - 4) = 0.$$

Thus  $t \in \{1, 2, 4\}$ , hence

$$\boxed{x \in \{1, 8, 64\}}.$$

### WORKED EXAMPLE 5 A reciprocal quartic that becomes a quadratic

For  $x > 0$ , solve

$$x^{1/2} + 5x^{-1/2} = 6.$$

#### SOLUTION

Put  $t = \sqrt{x} > 0$ . Then

$$t + \frac{5}{t} = 6.$$

Multiplying by  $t > 0$  gives

$$t^2 - 6t + 5 = 0 = (t - 1)(t - 5).$$

Hence  $t = 1$  or  $5$ , so

$$\boxed{x = 1 \text{ or } 25}.$$

**COMMON TRAP.** Multiplying by  $t$  is safe only because  $t = \sqrt{x} > 0$  has already been recorded. Domain first, algebra second.

### WORKED EXAMPLE 6 Minimum value from a power substitution

For  $x > 0$ , find the minimum value of

$$x^{2/3} + \frac{16}{x^{2/3}} + 2x^{1/3} + \frac{8}{x^{1/3}}.$$

#### SOLUTION

Let  $t = x^{1/3} > 0$ . Then the expression is

$$t^2 + \frac{16}{t^2} + 2t + \frac{8}{t}.$$

Group it as

$$\left(t - \frac{4}{t}\right)^2 + 8 + 2\left(t + \frac{4}{t}\right).$$

Now  $t + 4/t \geq 4$  by AM-GM, and the square is non-negative. Equality in both conditions occurs when  $t = 2$ . Thus the minimum is

$$0 + 8 + 2(4) = \boxed{16}.$$

### WORKED EXAMPLE 7 Fractional powers with a restricted inequality

Solve for  $x > 0$

$$x^{1/2} - 5x^{1/4} + 6 \leq 0.$$

#### SOLUTION

Set  $t = x^{1/4} > 0$ . Then

$$t^2 - 5t + 6 \leq 0 \iff (t - 2)(t - 3) \leq 0.$$

Hence  $2 \leq t \leq 3$ . Raising to the fourth power preserves order because all quantities are non-negative:

$$\boxed{16 \leq x \leq 81}.$$

### WORKED EXAMPLE 8 Counting integer parameter pairs from prime exponents

Let  $p, q$  be integers satisfying  $-5 \leq p, q \leq 5$ . How many ordered pairs  $(p, q)$  make

$$\frac{72^p 18^q}{12^{p+q}}$$

an integer?

#### SOLUTION

Prime-factorise:

$$72 = 2^3 3^2, \quad 18 = 2 \cdot 3^2, \quad 12 = 2^2 \cdot 3.$$

Thus

$$\frac{72^p 18^q}{12^{p+q}} = 2^{p-q} 3^{p+q}.$$

We need

$$p - q \geq 0, \quad p + q \geq 0.$$

Equivalently  $-p \leq q \leq p$ . This is possible only for  $p \geq 0$ . For  $p = 0, 1, 2, 3, 4, 5$  there are respectively

$$1, 3, 5, 7, 9, 11$$

choices of  $q$ . Therefore the number of ordered pairs is

$$1 + 3 + 5 + 7 + 9 + 11 = \boxed{36}.$$

**TMUA SHORTCUT.** After simplification, an integer-condition question can become a lattice-point count between two straight lines.

### WORKED EXAMPLE 9 Three parameters and three prime-exponent conditions

For integers  $m, n, p$ , determine exactly when

$$E = \frac{12^{m+n} 15^{n+p}}{18^n 10^p 6^m}$$

is an integer.

#### SOLUTION

Prime-factorise every base:

$$12 = 2^2 \cdot 3, \quad 15 = 3 \cdot 5, \quad 18 = 2 \cdot 3^2, \quad 10 = 2 \cdot 5, \quad 6 = 2 \cdot 3.$$

Collect the exponent of each prime separately:

$$E = 2^{2(m+n)-n-p-m} 3^{m+n+n+p-2n-m} 5^{n+p-p}.$$

Hence

$$E = 2^{m+n-p} 3^p 5^n.$$

Therefore  $E$  is an integer precisely when

$$\boxed{n \geq 0, \quad p \geq 0, \quad m + n - p \geq 0}.$$

Equivalently,

$$\boxed{n \geq 0, \quad p \geq 0, \quad p \leq m + n}.$$

**KEY IDEA.** With three parameters, the answer is often a region cut out by several linear inequalities. The algebra is finished as soon as every composite base has disappeared.

**WORKED EXAMPLE 10** When integrality forces all three parameters to coincide

For integers  $m, n, p$ , determine when

$$F = \frac{8^m 27^n 125^p}{8^n 27^p 125^m}$$

is an integer.

**SOLUTION**

Rewrite in prime powers:

$$F = 2^{3(m-n)} 3^{3(n-p)} 5^{3(p-m)}.$$

For  $F$  to be an integer, all three exponents must be non-negative:

$$m - n \geq 0, \quad n - p \geq 0, \quad p - m \geq 0.$$

Thus

$$m \geq n \geq p \geq m.$$

The only way a chain of inequalities can return to its starting point is equality throughout:

$$\boxed{m = n = p}.$$

**TMUA SHORTCUT.** Cyclic exponent conditions such as  $m \geq n \geq p \geq m$  collapse immediately to equality. Spotting this can save a large amount of casework.

**WORKED EXAMPLE 11** Counting triples after the prime-exponent reduction

Suppose  $m, n, p \in \{0, 1, 2, 3, 4\}$ . How many ordered triples make

$$\frac{12^{m+n} 15^{n+p}}{18^n 10^p 6^m}$$

an integer?

**SOLUTION**

From the previous worked example, the condition is

$$p \leq m + n$$

because  $n, p \geq 0$  automatically in the stated range.

There are  $5^3 = 125$  triples in total. Count the failures  $p > m + n$ .

For a fixed value  $s = m + n$ , the number of pairs  $(m, n)$  with  $0 \leq m, n \leq 4$  is

$$1, 2, 3, 4, 5, 4, 3, 2, 1$$

for  $s = 0, 1, \dots, 8$ . Only  $s = 0, 1, 2, 3$  can fail because  $p \leq 4$ . The number of forbidden  $p$  values is respectively

$$4, 3, 2, 1.$$

Hence the number of failures is

$$1(4) + 2(3) + 3(2) + 4(1) = 20.$$

Therefore the required number of triples is

$$\boxed{125 - 20 = 105}.$$