

CF MATHS

TMUA PAPER 1

DIFFERENTIATION

Teacher Notes • Tangents • Normals • Stationary Points • Optimisation • Mixed Synthesis

SPECIFICATION NOTE. Aligned to the current UAT-UK TMUA content specification for the 2026–27 admissions cycle, especially MM6.1–MM6.3. The core is derivative as tangent gradient/rate of change, first and second derivatives, rational-power differentiation after algebraic simplification, tangents and normals, stationary maxima/minima, and strictly increasing/decreasing functions. Differentiation from first principles is excluded; points of inflexion are not directly examined.

DESIGN OF THIS EDITION. Worked-example headings carry no difficulty labels. The mathematics becomes progressively more demanding. The derivative itself remains syllabus-level; the challenge comes from algebra, inequalities, cubics and quartics, root counts, tangent/normal geometry, exact optimisation, and multi-topic information that must be converted into a differentiable polynomial or rational-power expression.

Prepared for CF Maths • 93 fully worked examples • extensive graph visualisation • 60-question TMUA-style review

1. Derivative as Gradient and Rate of Change

The derivative is not merely a rule for reducing powers. At a point $x = a$, $f'(a)$ is the gradient of the tangent to $y = f(x)$. In an applied setting it is the instantaneous rate at which the output changes with respect to the input.

Core theory

For rational n ,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

TMUA may require substantial algebraic simplification before this rule becomes visible. Products such as $(x+1)^2x^{-1/2}$ should normally be expanded into a sum of powers first. A quotient should often be rewritten using negative indices. The specification does not require product, quotient or chain rules.

The second derivative $f''(x)$ is the derivative of $f'(x)$. At a stationary point $x = a$ where $f'(a) = 0$, a useful classification is

$$f''(a) > 0 \Rightarrow \text{local minimum}, \quad f''(a) < 0 \Rightarrow \text{local maximum}.$$

If $f''(a) = 0$, no conclusion follows without further reasoning.

HIGH-VALUE STRUCTURE. Before differentiating: (i) expand, factor, cancel or rewrite powers; (ii) state the domain; (iii) decide what the question actually asks. The derivative may be only the bridge to an inequality, a root count, a tangent condition, a minimum distance, or a parameter range.

COMMON TRAPS. $f'(a) = 0$ means “stationary”, not automatically “maximum”. A normal gradient is $-1/m$ only when the tangent gradient $m \neq 0$. On a closed interval, global extrema require endpoints as well as stationary points. When a rational power restricts the domain, do not solve derivative equations outside that domain.

WORKED EXAMPLE 1 Simplify first: a disguised sum of rational powers

For $x > 0$, differentiate

$$f(x) = \frac{(2x+1)^2}{\sqrt{x}}.$$

SOLUTION

Expand before differentiating:

$$f(x) = \frac{4x^2 + 4x + 1}{x^{1/2}} = 4x^{3/2} + 4x^{1/2} + x^{-1/2}.$$

Hence

$$f'(x) = 6x^{1/2} + 2x^{-1/2} - \frac{1}{2}x^{-3/2}.$$

A useful single-fraction form is

$$f'(x) = \frac{12x^2 + 4x - 1}{2x^{3/2}}.$$

SHORTCUT. Expand into powers before differentiating. This keeps the method entirely within MM6.2.

WORKED EXAMPLE 2 A derivative inequality with a square-root domain

For $x > 0$, let

$$f(x) = \sqrt{x}(x - 2)^2.$$

Find all x for which $f'(x) > 0$.

SOLUTION

Expand:

$$f(x) = x^{5/2} - 4x^{3/2} + 4x^{1/2}.$$

Then

$$f'(x) = \frac{5}{2}x^{3/2} - 6x^{1/2} + 2x^{-1/2} = \frac{5x^2 - 12x + 4}{2\sqrt{x}}.$$

The denominator is positive for $x > 0$, so the sign is controlled by

$$5x^2 - 12x + 4 = (5x - 2)(x - 2).$$

Thus

$$0 < x < \frac{2}{5} \quad \text{or} \quad x > 2.$$

TRAP. Do not multiply an inequality by an expression until you know its sign. Here $2\sqrt{x} > 0$ because the domain is $x > 0$.

WORKED EXAMPLE 3 A global minimum from reciprocal powers

For $x > 0$, find the minimum value of

$$f(x) = \frac{x^3}{3} - 3x + \frac{4}{x}.$$

SOLUTION

Differentiate:

$$f'(x) = x^2 - 3 - \frac{4}{x^2} = \frac{x^4 - 3x^2 - 4}{x^2}.$$

Set $u = x^2 > 0$:

$$u^2 - 3u - 4 = (u - 4)(u + 1) = 0.$$

Hence $x = 2$ is the only stationary point in the domain. For $0 < x < 2$, $f'(x) < 0$; for $x > 2$, $f'(x) > 0$, so this is a minimum. Its value is

$$f(2) = \frac{8}{3} - 6 + 2 = -\frac{4}{3}.$$

Therefore

$$f_{\min} = -\frac{4}{3}.$$

KEY IDEA. When reciprocal powers appear, multiplying the derivative equation by a positive power of x often produces a polynomial in x^2 .

WORKED EXAMPLE 4 Closed-interval extrema with a cubic derivative

On $-3 \leq x \leq 5$, find the maximum and minimum values of

$$F(x) = \frac{x^4}{4} - x^3 - 3x^2 + 8x.$$

SOLUTION

Differentiate and factor:

$$F'(x) = x^3 - 3x^2 - 6x + 8 = (x + 2)(x - 1)(x - 4).$$

The stationary x -values in the interval are $-2, 1, 4$. For a closed interval, also check $x = -3$ and $x = 5$:

$$F(-3) = -\frac{15}{4}, \quad F(-2) = -16, \quad F(1) = \frac{17}{4}, \quad F(4) = -16, \quad F(5) = -\frac{15}{4}.$$

Hence

$$F_{\max} = \frac{17}{4} \text{ at } x = 1, \quad F_{\min} = -16 \text{ at } x = -2, 4.$$

EXAM TECHNIQUE. A derivative sign chart tells you where to look, but a closed-interval global maximum/minimum still requires endpoint evaluation.

WORKED EXAMPLE 5 A parameter chosen so a polynomial is increasing everywhere

Find all real k for which

$$f(x) = x^3 + 3kx^2 + 3x$$

is strictly increasing for every real x .

SOLUTION

We need $f'(x) > 0$ for every real x :

$$f'(x) = 3x^2 + 6kx + 3 = 3(x^2 + 2kx + 1).$$

For a quadratic with positive leading coefficient to be strictly positive for all real x , its discriminant must be negative:

$$(2k)^2 - 4 < 0 \implies k^2 < 1.$$

Therefore

$$-1 < k < 1.$$

TRAP. Using $\Delta \leq 0$ would allow a repeated root of f' , where $f'(x) = 0$. The question asks for strictly increasing, so we need $f'(x) > 0$.

WORKED EXAMPLE 6 Reconstructing a cubic from its derivative

A cubic f satisfies

$$f'(x) = 3(x-1)(x-4), \quad f(0) = 7.$$

Find $f(x)$.

SOLUTION

Expand the derivative:

$$f'(x) = 3x^2 - 15x + 12.$$

An antiderivative polynomial is

$$f(x) = x^3 - \frac{15}{2}x^2 + 12x + C.$$

Using $f(0) = 7$ gives $C = 7$. Hence

$$f(x) = x^3 - \frac{15}{2}x^2 + 12x + 7.$$

KEY IDEA. Although this uses the inverse relationship between differentiation and integration, the algebraic idea is simple: matching a derivative determines the polynomial up to an additive constant.

WORKED EXAMPLE 7 Equal gradients at two points

Let

$$f(x) = x^4 - 4x^3 + kx^2.$$

Find k if the tangent gradients at $x = 1$ and $x = 3$ are equal.

SOLUTION

Differentiate:

$$f'(x) = 4x^3 - 12x^2 + 2kx.$$

Then

$$f'(1) = 4 - 12 + 2k = 2k - 8,$$

while

$$f'(3) = 108 - 108 + 6k = 6k.$$

Equating gives

$$2k - 8 = 6k \implies k = -2.$$

Thus

$$k = -2.$$

WORKED EXAMPLE 8 A stationary point that is not a turning point

Find and classify every stationary point of

$$f(x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 7.$$

SOLUTION

Recognise

$$f(x) = (x - 1)^5 + 8.$$

Therefore

$$f'(x) = 5(x - 1)^4.$$

The only stationary point is $(1, 8)$. But $f'(x) \geq 0$ on both sides of $x = 1$, so the derivative does not change sign. The curve continues increasing through the point.

$(1, 8)$ is stationary but is neither a local maximum nor a local minimum.

TRAP. $f'(x) = 0$ identifies stationary points. Classification requires a sign change or other valid argument.

WORKED EXAMPLE 9 Rational powers plus a parameter condition

For $x > 0$, define

$$f(x) = x^{5/2} - ax^{3/2} + 4x^{1/2}.$$

Find all a for which $x = 1$ is stationary.

SOLUTION

Differentiate:

$$f'(x) = \frac{5}{2}x^{3/2} - \frac{3a}{2}x^{1/2} + 2x^{-1/2}.$$

At $x = 1$,

$$f'(1) = \frac{5}{2} - \frac{3a}{2} + 2 = \frac{9 - 3a}{2}.$$

Stationarity requires $f'(1) = 0$, so $a = 3$.

$$a = 3.$$

WORKED EXAMPLE 10 A hidden cubic in the derivative equation

For $x > 0$, let

$$f(x) = \frac{x^4 - 4x + 3}{x}.$$

Find all stationary points.

SOLUTION

Simplify first:

$$f(x) = x^3 - 4 + 3x^{-1}.$$

Then

$$f'(x) = 3x^2 - 3x^{-2} = \frac{3(x^4 - 1)}{x^2}.$$