

CF MATHS

TMUA PAPER 1

EXPONENTIALS & LOGARITHMS

Teacher Notes • Expanded Theory • Worked Examples • Synthesis

SPECIFICATION NOTE. Aligned to the current UAT-UK TMUA content specification for the 2026–27 admissions cycle. Core coverage is MM5.1–MM5.3: graphs of $y = a^x$ for positive bases, logarithm laws, and exponential equations reducible by algebraic manipulation. Graph work also overlaps with MM8. The published specification explicitly states that questions requiring the change-of-base formula will not be set, so these notes do not rely on it.

DESIGN OF THIS EDITION. The chapter is intentionally above routine school-drill level. Worked examples have no difficulty labels; instead, complexity rises through hidden substitutions, domain restrictions, simultaneous systems, polynomial reduction, parameter/root-count questions, graphical reasoning, and mixed-topic synthesis. The final bank is TMUA-style multiple choice.

Prepared for CF Maths • Original questions and fully worked solutions

Core Theory and Graph Structure

What the specification requires - and what TMUA actually tests

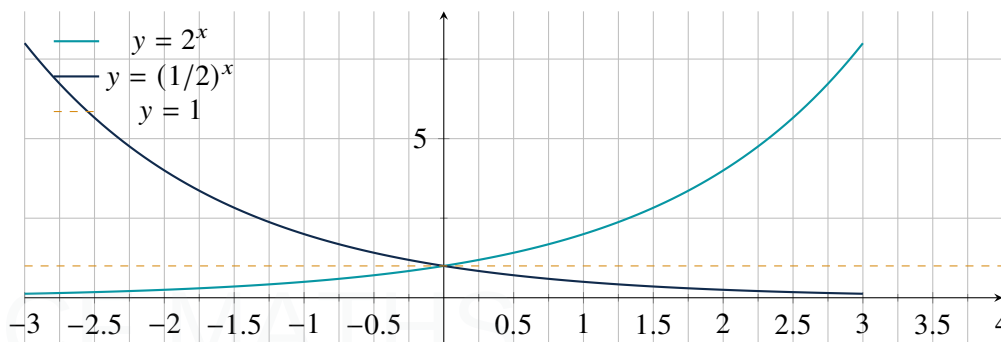
For a fixed positive base a , the exponential function is $y = a^x$. The logarithm $y = \log_a x$ is the inverse relationship defined by

$$a^b = c \iff b = \log_a c,$$

with $a > 0$, $a \neq 1$, and $c > 0$. The syllabus content is short; the admissions-test challenge is usually the *algebra surrounding it*.

Essential graph facts.

- If $a > 1$, a^x is strictly increasing; if $0 < a < 1$, it is strictly decreasing.
- Every exponential graph passes through $(0, 1)$ and has horizontal asymptote $y = 0$.
- $a^x > 0$ for every real x . This positivity condition is crucial after a substitution such as $t = a^x$.
- $\log_a x$ has domain $x > 0$, passes through $(1, 0)$, and has vertical asymptote $x = 0$.
- For $a > 1$, $\log_a x$ is increasing; for $0 < a < 1$, it is decreasing.
- $y = a^x$ and $y = \log_a x$ are reflections of each other in $y = x$.



Logarithm laws and domain discipline

For $x, y > 0$,

$$\log_a x + \log_a y = \log_a(xy), \quad \log_a x - \log_a y = \log_a\left(\frac{x}{y}\right), \quad k \log_a x = \log_a(x^k).$$

The domain check comes *before* simplification. For example,

$$\log_2(x-1) + \log_2(x+1)$$

requires $x > 1$, even though the product $(x-1)(x+1) = x^2 - 1$ is positive also when $x < -1$.

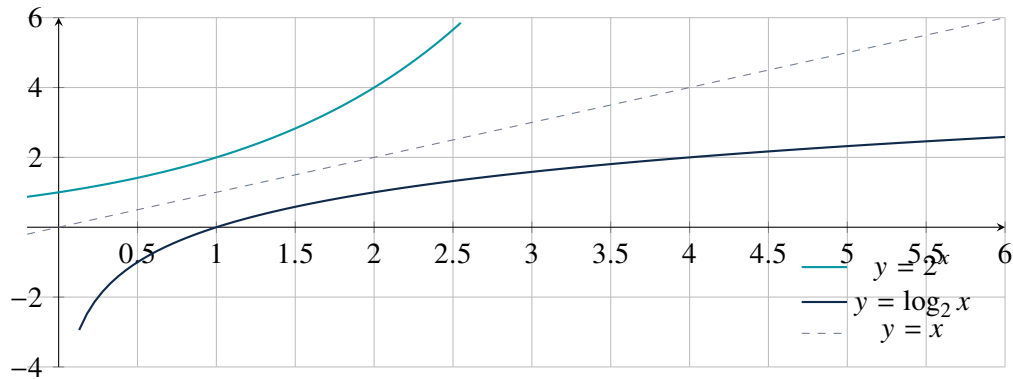
DOMAIN TRAP. Algebraically equivalent-looking expressions need not have the same logarithmic domain. Never recover an excluded root simply because it satisfies the polynomial obtained after combining logs.

The master substitutions

- If powers occur as $a^x, a^{2x}, a^{3x}, \dots$, set $t = a^x > 0$.
- If reciprocal powers occur, set $t = a^x > 0$ and use $a^{-x} = 1/t$, or use $u = a^x + a^{-x}$ when symmetry is visible.
- If powers are in a common multiple, rewrite first: $8^x = (2^x)^3$, $9^x = (3^x)^2$.

- If logarithms share a base, combine them before expanding.
- If a variable appears as a logarithm, a substitution such as $u = \log_2 x$ often turns the equation into a quadratic or cubic.

TMUA SHORTCUT. After a substitution, solve the algebraic problem completely in the new variable, apply its domain restriction ($t > 0$ or a logarithmic domain), and only then convert back to x .



CF MATHS

Exponential Functions: Transformations, Intersections and Structural Reasoning

The graph is not decoration. It tells you positivity, monotonicity, asymptotes, and how many solutions an equation can have. In parameter questions, sketch first whenever the algebra alone hides the number of intersections.

WORKED EXAMPLE 1 A translated exponential graph

For

$$f(x) = 3^{x-2} + 1,$$

state the horizontal asymptote, range, y-intercept, and the x-coordinate at which $f(x) = 4$.

SOLUTION

The basic graph 3^x is shifted two units right and one unit up. Hence the horizontal asymptote is

$$y = 1,$$

and because $3^{x-2} > 0$, the range is $y > 1$.

At $x = 0$,

$$f(0) = 3^{-2} + 1 = \frac{1}{9} + 1 = \frac{10}{9}.$$

Finally,

$$3^{x-2} + 1 = 4 \iff 3^{x-2} = 3 \iff x - 2 = 1,$$

so $x = 3$.

KEY IDEA. A shift changes the asymptote but never destroys positivity of the exponential part.

WORKED EXAMPLE 2 A reflected exponential and its intercept

The curve is

$$y = 4 - 2^{x+1}.$$

Find its range, horizontal asymptote, and both coordinate-axis intercepts.

SOLUTION

Since $2^{x+1} > 0$, we have $y < 4$, so the range is $(-\infty, 4)$ and the horizontal asymptote is $y = 4$.

At $x = 0$,

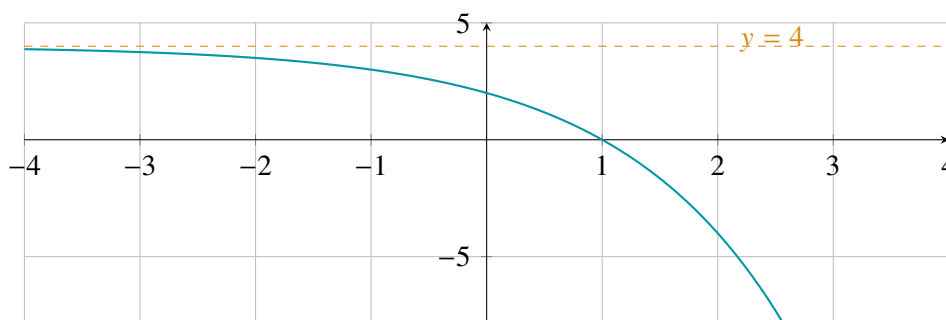
$$y = 4 - 2 = 2,$$

so the y-intercept is $(0, 2)$.

For the x-intercept,

$$4 - 2^{x+1} = 0 \iff 2^{x+1} = 4 = 2^2,$$

so $x = 1$. Therefore the x-intercept is $(1, 0)$.



WORKED EXAMPLE 3 Recovering a translation parameter

A curve has equation $y = 2^x + k$ and passes through $(3, 11)$. Find k , the horizontal asymptote, and the point where $x = 0$.

SOLUTION

Substitute $(3, 11)$:

$$11 = 2^3 + k = 8 + k,$$

so $k = 3$. Thus the curve is $y = 2^x + 3$, its asymptote is $y = 3$, and at $x = 0$,

$$y = 1 + 3 = 4.$$

Therefore the required point is $(0, 4)$.

WORKED EXAMPLE 4 Two exponentials meet after a reciprocal rewrite

Solve

$$2^x = 8 \cdot 2^{-x}.$$

SOLUTION

Multiply by $2^x > 0$:

$$2^{2x} = 8 = 2^3.$$

Hence $2x = 3$ and

$$x = \frac{3}{2}.$$

KEY IDEA. Multiplying by a positive exponential never changes equivalence.

WORKED EXAMPLE 5 Absolute value inside an exponent

Solve

$$3^{|x-1|} = 27.$$

SOLUTION

Since $27 = 3^3$ and the base is the same,

$$|x - 1| = 3.$$

Therefore

$$x - 1 = 3 \quad \text{or} \quad x - 1 = -3,$$

so

$$\boxed{x = 4 \text{ or } x = -2}.$$

WORKED EXAMPLE 6 A decreasing-base inequality

Solve

$$\left(\frac{1}{3}\right)^{2x-1} > 9.$$

SOLUTION

Write $9 = (1/3)^{-2}$. Because the base $1/3$ lies between 0 and 1, the exponential function is decreasing, so the inequality reverses when we compare exponents:

$$2x - 1 < -2.$$

Thus

$$2x < -1 \iff x < -\frac{1}{2}.$$

TRAP. For $0 < a < 1$, larger outputs correspond to smaller exponents. This is one of the most common TMUA sign traps.

WORKED EXAMPLE 7 A global minimum from reciprocal symmetry

Find the minimum value of

$$2^x + 2^{-x}$$

over all real x .**SOLUTION**

Set $t = 2^x > 0$. The expression becomes

$$t + \frac{1}{t}.$$

Since

$$\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 \geq 0,$$

we have

$$t + \frac{1}{t} \geq 2.$$

Equality occurs when $t = 1$, i.e. $2^x = 1$, so $x = 0$. Therefore the minimum value is

$$\boxed{2}.$$

KEY IDEA. The positivity $t > 0$ is what makes the square argument valid.