

CF MATHS

TMUA PAPER 1 GRAPHS OF FUNCTIONS

Teacher Notes • Transformations • Root Counts • Intersections • Modulus •
Logs/Exponentials • Trigonometry

SPECIFICATION NOTE. Aligned to the current UAT-UK TMUA Graphs of Functions content, especially MM8.1–MM8.7: recognition and sketching of common graphs; transformations and composition; the effect of parameters in lines and quadratics; differentiation to determine shape; algebraic intercepts and real-root counts; and the geometric interpretation of equations as graph intersections. Floor/ceiling graphs are included as a clearly marked CF Maths extension because the user-requested staircase reasoning is useful but is not named in MM8.

DESIGN OF THIS EDITION. The chapter is graph-led. Worked-example headings carry no difficulty labels. Whenever a sketch genuinely clarifies the reasoning, the sketch appears before or inside the solution. The target skill is not artistic graph drawing: it is extracting the information needed to count roots, locate parameter thresholds, detect tangency, control domains, and switch between algebra and geometry quickly.

Prepared for CF Maths • 130 worked examples • graph-first intersection reasoning • 60-question
TMUA-style review

1. Graphing Language: Shape Before Calculation

A graph question is often an equation question in disguise. Before solving anything, decide what each side can look like, where it is defined, and how many intersections are even possible.

Core theory

The key equivalence is

$$f(x) = g(x) \iff \text{the graphs } y = f(x) \text{ and } y = g(x) \text{ intersect.}$$

A root of $f(x) = 0$ is an x -intercept. A repeated root is a contact with the axis: the curve may touch rather than cross. A point where two graphs merely approach an asymptote is *not* an intersection.

For polynomials, end behaviour comes from the degree and the sign of the leading coefficient. Odd degree means opposite ends; even degree means the same direction at both ends. For non-polynomial graphs, domain and range can rule out intersections before any algebra.

GRAPH-FIRST ROUTINE. (1) State domains. (2) Mark intercepts/asymptotes/turning points. (3) Sketch only the features that can change the intersection count. (4) Use algebra only after the possible number of solutions is clear.

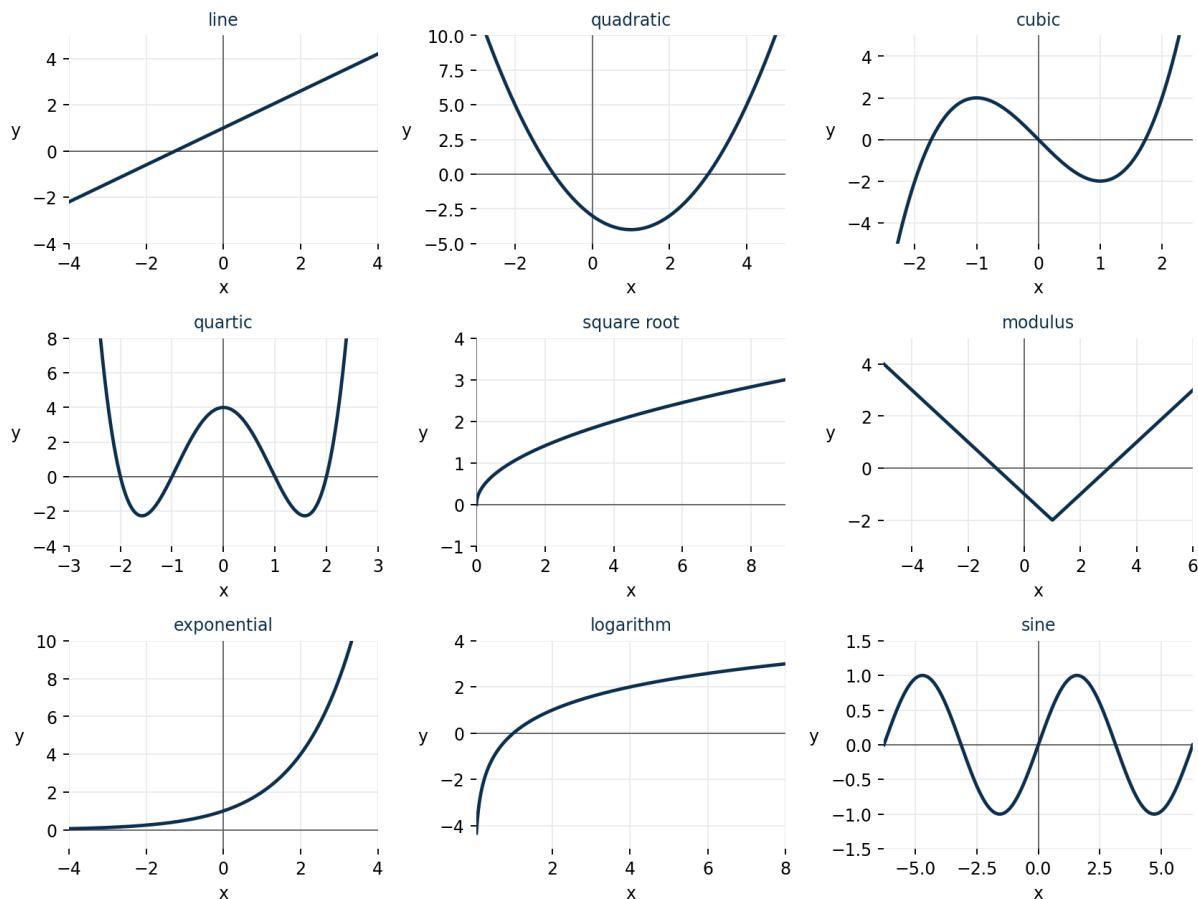
COMMON TRAP. A sketch is evidence about structure, not permission to invent exact coordinates. If the question asks for exact roots, use the sketch to choose the algebraic route, then solve exactly.

WORKED EXAMPLE 1 The family of common TMUA shapes

Match the nine graphs in the gallery to the corresponding function families and state one feature that immediately distinguishes each family.

SOLUTION

Lines have constant gradient; quadratics have one axis of symmetry; cubics have opposite end behaviour; quartics have the same end behaviour at both ends; square-root graphs have a one-sided domain; modulus graphs contain reflected pieces or sharp corners; exponentials stay positive; logarithms have a vertical asymptote; sine is bounded and periodic.



KEY IDEA. Learn shapes by invariants: domain, range, symmetry, asymptotes, periodicity and end behaviour. Those survive many transformations.

WORKED EXAMPLE 2 A quadratic from structural information

Sketch $y = x^2 - 4x - 5$ without making a table of values. State its axis, vertex and intercepts.

SOLUTION

Complete the square:

$$y = (x - 2)^2 - 9.$$

So the axis is $x = 2$ and the vertex is $(2, -9)$. Also

$$x^2 - 4x - 5 = (x - 5)(x + 1),$$

so the x -intercepts are -1 and 5 , while the y -intercept is -5 . These four facts determine the useful sketch.

SHORTCUT. For a quadratic, vertex form controls shape and factor form controls roots. Use both forms rather than expanding repeatedly.

WORKED EXAMPLE 3 A factored cubic

Sketch the essential shape of $y = (x + 2)(x - 1)(x - 3)$ and determine the sign of the function on each interval cut by its roots.

SOLUTION

The roots are $-2, 1, 3$, all simple, so the graph crosses the axis at each. The leading coefficient is positive, so the cubic goes from bottom-left to top-right. The sign alternates at each simple root:

$$(-\infty, -2) : -; \quad (-2, 1) : +; \quad (1, 3) : -; \quad (3, \infty) : +.$$

No derivative is required for this qualitative sketch.

KEY IDEA. Simple roots force a sign change; even-multiplicity roots do not.

WORKED EXAMPLE 4 Repeated roots and contact

Compare the graphs of $y = (x - 1)^2(x + 2)$ and $y = (x - 1)(x + 2)^2$ near their roots.

SOLUTION

In the first function, $x = 1$ is a double root, so the graph touches the axis there, while $x = -2$ is simple and the graph crosses. In the second, the roles reverse. Multiplicity is visible geometrically: odd multiplicity crosses; even multiplicity touches.

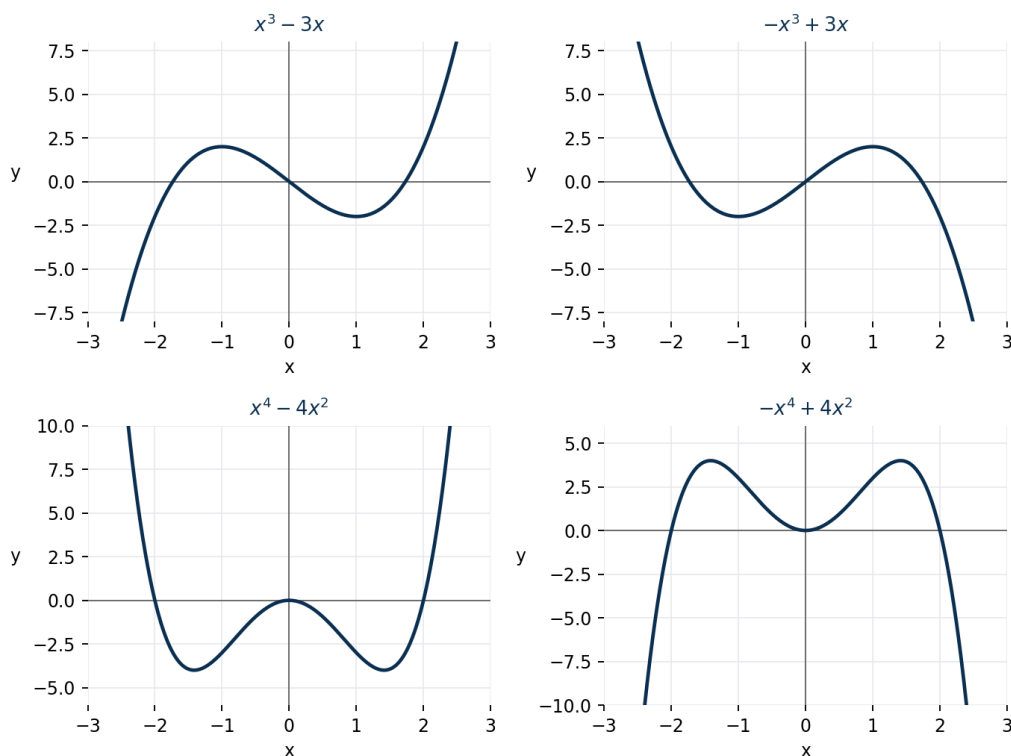
TRAP. Distinct roots and roots counted with multiplicity are different questions. TMUA wording often asks for distinct real solutions.

WORKED EXAMPLE 5 Polynomial end behaviour

Without solving any equations, explain how the four curves in the gallery differ at their ends and what that tells you about their possible number of real roots.

SOLUTION

A positive-leading cubic goes down-left/up-right; a negative-leading cubic does the reverse. A positive-leading quartic rises at both ends; a negative-leading quartic falls at both ends. Therefore every real cubic has at least one real root, while a quartic can have none. A quartic can have 0, 1, 2, 3, or 4 distinct real roots depending on tangencies and crossings.



WORKED EXAMPLE 6 A square-root domain

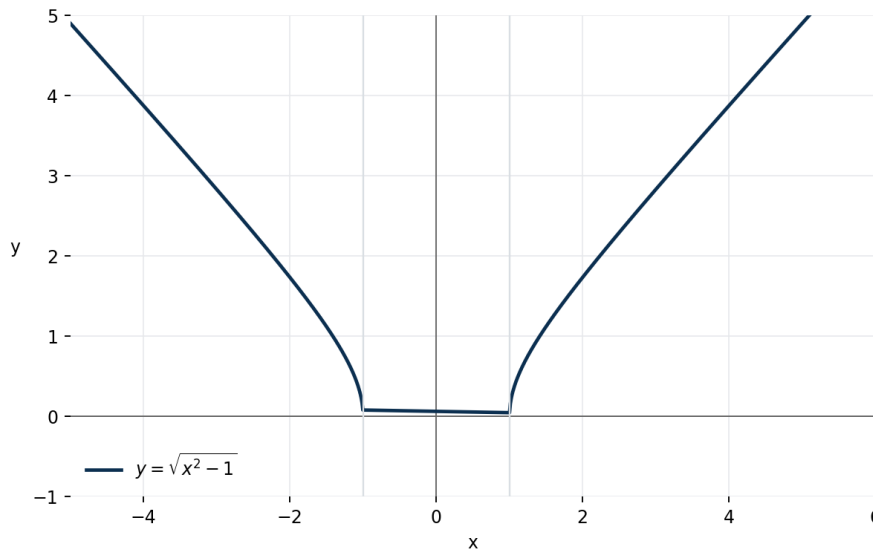
Sketch $y = \sqrt{x^2 - 1}$. Where is the graph defined, and why is there a gap rather than a vertical asymptote?

SOLUTION

Real values require

$$x^2 - 1 \geq 0 \iff x \leq -1 \text{ or } x \geq 1.$$

At $x = \pm 1$ the function equals 0, so these endpoints belong to the graph. There is simply no real graph for $-1 < x < 1$; nothing tends to infinity there, so these are not vertical asymptotes.



DOMAIN FIRST. A missing interval caused by a square-root restriction is fundamentally different from an asymptote.

WORKED EXAMPLE 7 An exponential and its asymptote

Sketch $y = 3^{x-1} - 2$. State its horizontal asymptote, range and y -intercept.

SOLUTION

Start with $y = 3^x$. Shift right by 1 and down by 2. Hence the horizontal asymptote is $y = -2$ and the range is $(-2, \infty)$. At $x = 0$,

$$y = 3^{-1} - 2 = -\frac{5}{3}.$$

The graph is increasing because the base 3 exceeds 1.

TRAP. A horizontal asymptote describes limiting behaviour; it is not automatically a forbidden y -value after every transformation. Here, however, $3^{x-1} > 0$, so $y = -2$ really is never reached.

WORKED EXAMPLE 8 A logarithmic graph from its domain

Sketch $y = \log_2(x + 3) - 1$. State the vertical asymptote and the x -intercept.

SOLUTION

The input must satisfy $x + 3 > 0$, so $x > -3$ and the vertical asymptote is $x = -3$. For the intercept,

$$\log_2(x + 3) - 1 = 0 \iff x + 3 = 2 \iff x = -1.$$

The graph is increasing because the base 2 is greater than 1.

WORKED EXAMPLE 9 A periodic graph needs an interval

How many solutions does $\sin x = 0$ have? Why is the question incomplete unless an interval is supplied?

SOLUTION

The solutions are $x = n\pi$ for every integer n , so there are infinitely many. A trigonometric graph repeats indefinitely. A finite solution count requires a stated interval, such as $0 \leq x < 2\pi$, where the solutions are 0 and π .

EXAM HABIT. On periodic equations, read the interval before doing any algebra.

WORKED EXAMPLE 10 Sketching a complicated polynomial from factors

For

$$f(x) = -(x + 2)^2(x - 1)(x - 4),$$

sketch the useful features and state the sign of f on each interval.

SOLUTION

The double root -2 is a touch; the simple roots $1, 4$ are crossings. The degree is 4 with negative leading coefficient, so both ends go downward. At $x = 0$,

$$f(0) = -(2)^2(-1)(-4) = -16 < 0.$$

A double root does not change sign, so the graph remains below the axis through $x = -2$. It crosses at $x = 1$, becomes positive on $(1, 4)$, then crosses again at $x = 4$ and is negative for $x > 4$. Therefore

$$f(x) < 0 \text{ on } (-\infty, 1) \cup (4, \infty), \quad f(x) > 0 \text{ on } (1, 4).$$

TRAP. A sign chart is safer than trusting a rough sketch, especially when a repeated root is present.

2. Transformations and Composition

TMUA transformations are simple individually but become difficult when several are composed. Horizontal transformations act inside the function and therefore often feel reversed.

Core theory

From $y = f(x)$:

$$y = af(x) \quad \text{vertical scale/reflection,} \quad y = f(x) + a \quad \text{vertical shift,}$$

$$y = f(x+a) \quad \text{horizontal shift by } -a, \quad y = f(ax) \quad \text{horizontal scale by factor } 1/|a| \text{ and reflection if } a < 0.$$

For composition, $f(g(x))$ means apply g first. Keep domain restrictions from both stages.