

CF MATHS

TMUA PAPER 1

INTEGRATION

Teacher Notes • Signed Area • Modulus • FTC • Trapezium Rule • Differential Equations • Mixed Synthesis

SPECIFICATION NOTE. Aligned to the current UAT-UK TMUA integration content for the 2026–27 admissions cycle, especially MM7.1–MM7.6: definite integrals and signed area, integration of rational powers after algebraic simplification, the Fundamental Theorem of Calculus, combining integrals, the trapezium rule including over/under-estimation, and differential equations of the form $dy/dx = f(x)$. Floor/ceiling integrals are retained as a clearly marked CF Maths extension rather than presented as core MM7 content.

DESIGN OF THIS EDITION. Worked-example headings carry no difficulty labels. The examples become progressively more demanding. The technical antiderivatives remain syllabus-level; the difficulty comes from deciding where to split an area, handling modulus, finding hidden roots, choosing a useful symmetry, interpreting an accumulated-area function, controlling trapezium-rule error, and switching cleanly between integration and algebra, geometry, sequences, logs, exponentials or differentiation.

Prepared for CF Maths • extensive worked-example edition • graph-led area reasoning • 60-question TMUA-style review

1. Algebraic Integration: Structure Before Antiderivatives

Integration in TMUA is technically simple but structurally demanding. The power rule is only useful after the expression has been rewritten into a sum of powers.

Core theory

For rational $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

For a definite integral,

$$\int_a^b f(x) dx = F(b) - F(a), \quad F'(x) = f(x).$$

The integral is *signed*: parts below the x -axis contribute negatively. Geometric area is always non-negative and may require splitting at every root.

HIGH-VALUE HABITS. Simplify first. Expand products, cancel common factors only with the relevant exclusions retained, rewrite radicals as rational powers, and use symmetry before calculating. If a parameter changes the roots or sign pattern, determine the parameter regime before writing a single area integral.

COMMON TRAPS. Do not confuse $\int_a^b f(x) dx$ with the area between $y = f(x)$ and the axis. Do not integrate $|f(x)|$ as if the modulus were decorative. Do not apply the power rule to x^{-1} ; logarithmic antiderivatives are not part of the core rational-power integration technique used in this chapter.

WORKED EXAMPLE 1 A rational-power expression hiding a polynomial

Evaluate exactly

$$I = \int_1^4 \frac{(2x-3)^2(x+1)}{x^{3/2}} dx.$$

SOLUTION

Expand first:

$$(2x-3)^2(x+1) = (4x^2 - 12x + 9)(x+1) = 4x^3 - 8x^2 - 3x + 9.$$

Hence

$$I = \int_1^4 \left(4x^{3/2} - 8x^{1/2} - 3x^{-1/2} + 9x^{-3/2} \right) dx.$$

Therefore

$$I = \left[\frac{8}{5}x^{5/2} - \frac{16}{3}x^{3/2} - 6x^{1/2} - 18x^{-1/2} \right]_1^4 = \boxed{\frac{229}{15}}.$$

SHORTCUT. The apparent quotient is not a quotient-rule problem. Divide each term by $x^{3/2}$ and integrate powers.

WORKED EXAMPLE 2 A perfect-square cancellation hidden inside rational powers

For $x > 0$, evaluate

$$\int_1^9 \frac{(x^{3/2} - x^{1/2})^2}{x} dx.$$

SOLUTION

Factor before expanding:

$$x^{3/2} - x^{1/2} = x^{1/2}(x - 1).$$

So

$$\frac{(x^{3/2} - x^{1/2})^2}{x} = \frac{x(x - 1)^2}{x} = (x - 1)^2.$$

Hence

$$\int_1^9 (x - 1)^2 dx = \left[\frac{(x - 1)^3}{3} \right]_1^9 = \boxed{\frac{512}{3}}.$$

KEY IDEA. Cancellation can turn an intimidating rational-power integral into an ordinary polynomial.

WORKED EXAMPLE 3 A parameter determined by an exact integral

Find k if

$$\int_0^1 (12x^2 - 6kx + k) dx = 2.$$

SOLUTION

Integrate term by term:

$$\int_0^1 (12x^2 - 6kx + k) dx = 4 - 3k + k = 4 - 2k.$$

Thus

$$4 - 2k = 2 \implies \boxed{k = 1}.$$

EXAM TECHNIQUE. When the parameter appears only in coefficients, integrate first and solve the resulting algebraic equation afterwards.

WORKED EXAMPLE 4 Symmetry removes all odd powers

Evaluate

$$\int_{-2}^2 (3x^5 - 4x^3 + 7x^2 + 5) dx.$$

SOLUTION

The odd part $3x^5 - 4x^3$ integrates to zero on $[-2, 2]$. Therefore

$$I = 2 \int_0^2 (7x^2 + 5) dx = 2 \left[\frac{7}{3}x^3 + 5x \right]_0^2 = 2 \left(\frac{56}{3} + 10 \right) = \boxed{\frac{172}{3}}.$$

KEY IDEA. On $[-a, a]$, odd terms vanish and even terms double. Use this before doing any antiderivative work.

WORKED EXAMPLE 5 A parameter inside the limits

For $a > 0$, suppose

$$\int_0^a (6x - 2a) dx = 24.$$

Find a .

SOLUTION

Treat a as constant while integrating in x :

$$\int_0^a (6x - 2a) dx = [3x^2 - 2ax]_0^a = 3a^2 - 2a^2 = a^2.$$

Hence $a^2 = 24$ and $a > 0$, so

$$a = 2\sqrt{6}.$$

TRAP. A parameter in the upper limit is still a constant with respect to the integration variable.

WORKED EXAMPLE 6 Recovering an unknown coefficient from two definite integrals

Let $f(x) = ax^2 + bx + 1$. Given

$$\int_0^1 f(x) dx = 3, \quad \int_0^2 f(x) dx = 10,$$

find a and b .

SOLUTION

The first condition gives

$$\frac{a}{3} + \frac{b}{2} + 1 = 3 \implies 2a + 3b = 12.$$

The second gives

$$\frac{8a}{3} + 2b + 2 = 10 \implies 4a + 3b = 12.$$

Subtracting gives $2a = 0$, so $a = 0$, and then $b = 4$. Thus

$$a = 0, b = 4.$$

KEY IDEA. Definite-integral data often becomes a simultaneous-equations problem in the coefficients.

WORKED EXAMPLE 7 A polynomial identity obtained from an integral condition

For every real t ,

$$\int_0^t (3x^2 + 2ax + b) dx = t^3 + 5t^2 - 7t.$$

Find a and b .

SOLUTION

Integrating the left side gives

$$t^3 + at^2 + bt.$$

Since the equality holds for every real t , compare coefficients:

$$a = 5, \quad b = -7.$$

Hence

$$(a, b) = (5, -7).$$

SHORTCUT. An equality valid for every upper limit is a polynomial identity, so coefficient comparison is immediate.

WORKED EXAMPLE 8 An integral inequality becomes a quadratic inequality

Find all real k such that

$$\int_0^2 (x^2 - kx + k) dx > 0.$$

SOLUTION

Compute the integral:

$$\frac{8}{3} - 2k + 2k = \frac{8}{3}.$$

It is positive independently of k . Therefore

$$\text{every real } k.$$

TRAP. Do not assume a parameter matters merely because it appears in the integrand. Cancellation can make the final condition parameter-free.

WORKED EXAMPLE 9 A hidden exact derivative structure without using substitution

Evaluate

$$\int_0^1 (12x^5 - 20x^3 + 8x) dx.$$

SOLUTION

Integrate directly:

$$[2x^6 - 5x^4 + 4x^2]_0^1 = 2 - 5 + 4 = \boxed{1}.$$

The coefficients were chosen so the antiderivative at $x = 1$ collapses strongly.

EXAM TECHNIQUE. Exact arithmetic structure matters in admissions tests. Do not approximate when the expression is clearly designed to cancel.

WORKED EXAMPLE 10 A parameter chosen so two integrals are equalFind k if

$$\int_0^k (3x^2 + 1) dx = \int_k^2 (3x^2 + 1) dx, \quad 0 < k < 2.$$

SOLUTION

The total integral from 0 to 2 is

$$[x^3 + x]_0^2 = 10.$$

Equal halves mean

$$\int_0^k (3x^2 + 1) dx = 5.$$

Thus

$$k^3 + k = 5.$$

The function $k^3 + k$ is strictly increasing, so there is a unique solution in $(0, 2)$. Since $k = \sqrt[3]{5}$ is not simplifiable here, the TMUA point is the uniqueness argument rather than a decimal answer.

$$k \text{ is the unique root of } k^3 + k - 5 = 0 \text{ in } (0, 2).$$

KEY IDEA. An exact answer can be a uniquely specified algebraic root when the question is about structure rather than decimal calculation.

WORKED EXAMPLE 11 A rational-power integral that becomes a quadratic in $\sqrt{x}$ Find $a > 0$ if

$$\int_1^a (x^{-1/2} + x^{1/2}) dx = \frac{20}{3}.$$

SOLUTION

Integrate:

$$\left[2\sqrt{x} + \frac{2}{3}x^{3/2} \right]_1^a = \frac{20}{3}.$$

At $x = 1$ the antiderivative has value $2 + 2/3 = 8/3$, so the antiderivative at $x = a$ must equal $28/3$:

$$2\sqrt{a} + \frac{2}{3}a^{3/2} = \frac{28}{3}.$$

Let $u = \sqrt{a} > 0$. Then

$$2u + \frac{2}{3}u^3 = \frac{28}{3} \implies u^3 + 3u - 14 = 0.$$

Since $u = 2$ is a root and the remaining quadratic has no positive root, $u = 2$, hence

$$a = 4.$$

KEY IDEA. After integrating rational powers, substitutions such as $u = \sqrt{a}$ can turn the boundary equation into a polynomial.