

CF MATHS

TMUA PAPER 2 PROOF & LOGIC

Teacher Notes • Logic • Proof • Counterexamples • Number Theory •
Geometry

SPECIFICATION NOTE. Built around the current UAT-UK Paper 2 reasoning headings Arg1–Arg4, Prf1–Prf5 and Err1–Err2: implications and logical language, necessary/sufficient conditions, quantifiers and negation, direct proof, proof by cases, contradiction, counterexamples, deductions and proof ordering, and identifying errors in purported proofs. Formal truth tables are not required.

EXPANDED FINAL EDITION. This edition contains 217 worked mathematical examples, including expanded Paper 2 function logic, integral reasoning, geometric necessary/sufficient conditions, engineered counterexamples, contrapositive number theory and long proof-error diagnosis. It finishes with 150 TMUA-style multiple-choice questions, including a final advanced proof-error extension.

Prepared for CF Maths • 217 worked examples • functions, integrals, geometry and proof errors • 150
final MCQs

Proof & Logic - Teacher Notes • Original CF Maths teaching material

How to use these notes

Paper 2 contains 20 multiple-choice questions in 75 minutes. The mathematics comes from the same specification as Paper 1, but the emphasis is different: a candidate may have to decide whether a condition is necessary, sufficient, equivalent, impossible, or merely plausible; identify the first invalid line of a proof; construct a counterexample; or complete a chain of reasoning. The current UAT-UK description says Paper 2 assesses the ability to deal with mathematical reasoning and simple ideas from elementary logic.

HIGH-VALUE PAPER 2 HABITS

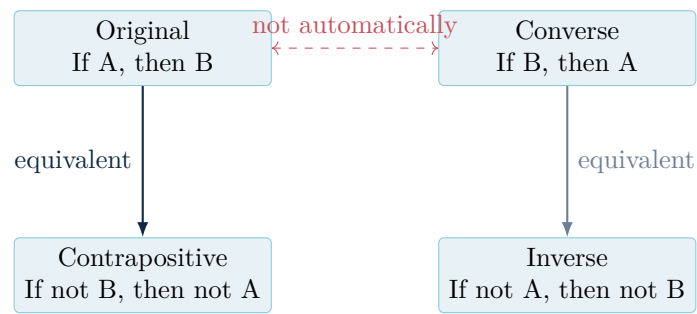
- Translate the sentence before doing algebra: what is the hypothesis, and what is the conclusion?
- For “necessary/sufficient”, test the two directions separately.
- For “must be true”, try to break each option with a counterexample before attempting a long proof.
- When negating, reverse quantifiers and change the final relation exactly.
- If a proof cancels, divides, squares, takes logs, or takes square roots, check the hidden conditions immediately.
- Small cases suggest a conjecture; they do not prove it.
- In an iff statement, proving one direction earns only half the logical job.

NOTATION. These notes occasionally use $A \Rightarrow B$, $A \Leftrightarrow B$ and $\nRightarrow$ as compact teacher shorthand. UAT-UK explicitly says candidates are not required to recognise or use formal symbolic notation for these logical terms in the test.

Specification map

Arg1	True/false; and/or/not; if, if, only if, iff; converse; contrapositive and their relationships.
Arg2	Necessary and sufficient conditions.
Arg3	For all, for some, there exists.
Arg4	Negating statements using the above language.
Prf1	Direct proof, proof by cases, contradiction, and disproof by counterexample.
Prf2–5	Deductions, conjectures from small cases, ordering proof statements, sophisticated chains of reasoning.
Err1–2	Identifying errors in purported proofs and common invalid deductions.

Visual map: implication, converse and contrapositive



CF MATHS

1. The Language of Mathematical Logic

Paper 2 is not testing abstract philosophy. It is testing whether a mathematical statement has been read with exact logical meaning. A statement is something that is either true or false. The words **and**, **or**, **not**, **if**, **only if** and **if and only if** change that meaning sharply.

Throughout these notes, $A \Rightarrow B$ is used only as teacher shorthand for “if A , then B ”. The current TMUA specification does *not* require candidates to use symbolic logic, and formal truth tables are not required.

HIGH-VALUE IDEAS

- TMUA uses inclusive *or*: “A or B” allows A, B, or both.
- “A if B” means: if B is true, then A is true.
- “A only if B” means: if A is true, then B must be true.
- “A if and only if B” means both directions are true.
- Negating a compound statement changes both the connective and the inequalities: $\text{not}(A \text{ and } B)$ becomes $(\text{not } A) \text{ or } (\text{not } B)$.

WORKED EXAMPLE 1

Truth of a compound divisibility statement

Let $n = 18$. Define

$$A : 3 \mid n, \quad B : 4 \mid n.$$

Determine the truth values of A , B , A and B , A or B , and not B .

SOLUTION

Since $18 = 3 \cdot 6$, statement A is true. Since $18/4$ is not an integer, B is false.

Therefore A and B is false because an “and” statement requires both parts to be true. The statement A or B is true because TMUA uses inclusive “or”: one true part is enough. Finally, not B is true because B is false.

So the truth pattern is

$$\boxed{T, F, F, T, T}.$$

TRAP. Do not read mathematical ‘or’ as ‘one or the other but not both’. It is inclusive unless the question explicitly says ‘exactly one’.

WORKED EXAMPLE 2

Inclusive or as a union of solution sets

Find all real x for which

$$x^2 = 4 \quad \text{or} \quad x = 2.$$

SOLUTION

The first statement gives $x = 2$ or $x = -2$. The second gives $x = 2$.

Because the logical connective is “or”, we take the union of these solution sets. Hence

$$\boxed{x \in \{-2, 2\}}.$$

The value $x = 2$ satisfying both statements is allowed; it is not excluded.

EXAM TECHNIQUE. For an OR statement, think union. For an AND statement, think intersection.

WORKED EXAMPLE 3

Reading 'A if B' in the correct direction

Consider the statement

$$“x^2 = 4 \text{ if } x = 2.”$$

Write it as an implication and decide whether it is true.

SOLUTION

The words “A if B” mean $B \Rightarrow A$. Here

$$B : x = 2, \quad A : x^2 = 4.$$

So the statement means

$$x = 2 \Rightarrow x^2 = 4.$$

This is true because substituting $x = 2$ gives $x^2 = 4$.

Notice that the statement does *not* say $x^2 = 4 \Rightarrow x = 2$; that converse is false because $x = -2$ also satisfies $x^2 = 4$.

TRAP. The word 'if' points from the phrase after it to the phrase before it.

WORKED EXAMPLE 4

Reading 'only if'

Decide whether the statement

“An integer n is divisible by 12 only if it is divisible by 4”

is true.

SOLUTION

“A only if B” means $A \Rightarrow B$. Thus we must test

$$12 \mid n \Rightarrow 4 \mid n.$$

If $12 \mid n$, then $n = 12k = 4(3k)$ for some integer k . Hence $4 \mid n$.

Therefore the statement is true.

KEY IDEA. 'Only if' introduces a condition that must hold whenever the first statement holds; it is therefore necessary.

WORKED EXAMPLE 5

An if-and-only-if parity statement

Show that for every integer n ,

$$n \text{ is even} \quad \text{if and only if} \quad n^2 \text{ is even.}$$

SOLUTION

An “if and only if” statement has two directions.

Forward direction. If n is even, write $n = 2k$. Then

$$n^2 = 4k^2 = 2(2k^2),$$

so n^2 is even.

Reverse direction. If n^2 is even, then n cannot be odd. Indeed, if $n = 2k + 1$, then

$$n^2 = 4k(k + 1) + 1,$$

which is odd. Hence n is even.

Both directions hold, so the biconditional is true.

TEACHER NOTE. Students often prove only the easier direction of an iff statement. Require them to name the two directions explicitly.

WORKED EXAMPLE 6

Negating a double inequality

Negate the statement

$$1 < x < 4.$$

SOLUTION

The statement means

$$x > 1 \quad \text{and} \quad x < 4.$$

To negate an AND statement, negate each part and replace AND by OR:

$$x \leq 1 \quad \text{or} \quad x \geq 4.$$

Therefore the exact negation is

$$\boxed{x \leq 1 \text{ or } x \geq 4}.$$

TRAP. The negation is not $1 \geq x \geq 4$, and strict inequalities must change to non-strict ones.

WORKED EXAMPLE 7

A compound statement with a parameter

For real k , consider

$$A : k > 2, \quad B : k^2 > 4.$$

Which implication is always true: $A \Rightarrow B$, $B \Rightarrow A$, both, or neither?

SOLUTION

If $k > 2$, then k is positive and squaring preserves the inequality against 2: $k^2 > 4$. Thus $A \Rightarrow B$ is true.

For the converse, $k = -3$ gives $k^2 = 9 > 4$, so B is true while A is false. Hence $B \Rightarrow A$ is false.

Therefore

$$\boxed{A \Rightarrow B \text{ only}}.$$

COUNTEREXAMPLE. A single value, here $k = -3$, is enough to destroy the converse.

WORKED EXAMPLE 8

Three logical conditions on a quadratic

Let

$$P(x) = x^2 - 2ax + a + 2.$$

Consider the statements

$$A : P \text{ has two distinct real roots}, \quad B : P \text{ has two positive roots}, \quad C : a > 2.$$

Determine which implications among A, B, C are always valid.