

CF MATHS

TMUA PAPER 1

SEQUENCES & SERIES

Teacher Notes • Expanded Worked-Example Edition

SPECIFICATION NOTE This edition is aligned to the current publicly available UAT-UK TMUA MM2.1–MM2.4 content: explicit and recurrent sequences, arithmetic progressions and series, finite and infinite geometric series with the convergence condition $|r| < 1$, sigma notation, factorial/binomial notation, and binomial expansions for positive integer powers. The mixed examples combine these ideas with other separately specified TMUA topics, but no extra technique is assumed without being developed in the notes.

DESIGN OF THIS EDITION The worked-example headings deliberately carry no difficulty labels. The questions themselves become progressively more demanding. The solutions emphasise structure, exact arithmetic and integer constraints rather than calculator-style approximation. Binomial theory has been expanded substantially, including coefficient systems that lead to two or three simultaneous equations. The final bank is written in TMUA-style multiple-choice form.

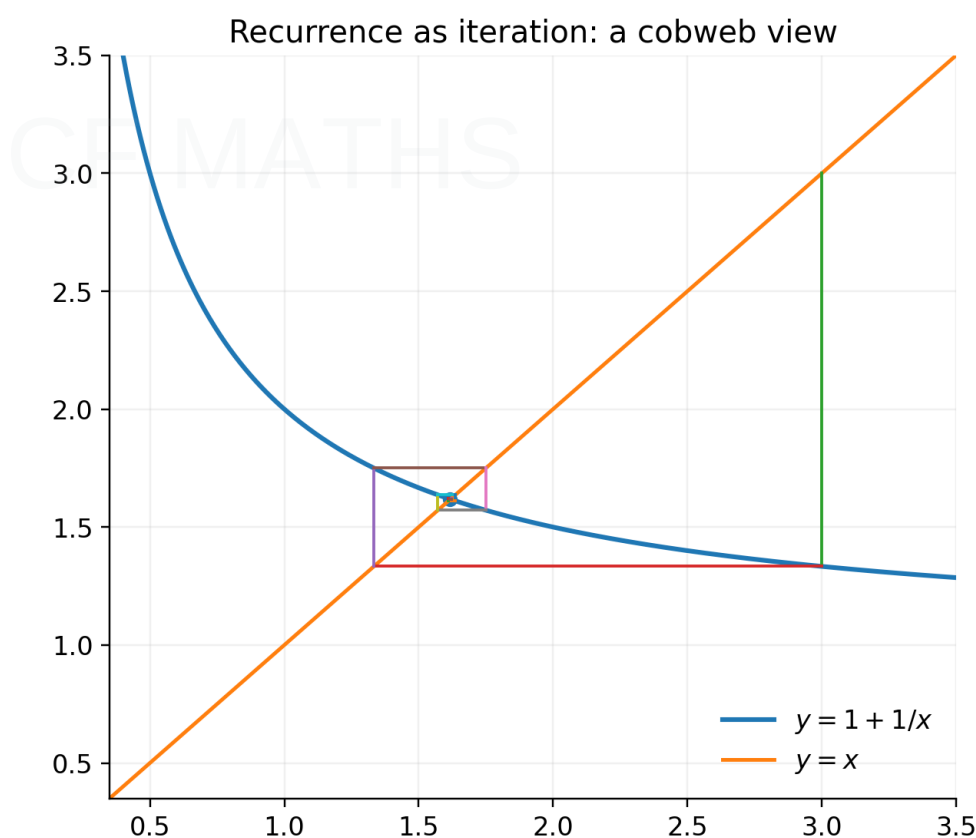
1. Sequences and Recurrence Relations

A *sequence* is an ordered list of terms. A *series* is the sum of terms of a sequence. An explicit formula such as $u_n = n^2 - 3n + 5$ allows direct substitution, whereas a recurrence defines later terms from earlier ones.

SYLLABUS BOUNDARY. TMUA requires students to work with explicit and recurrent sequences and with the sum to infinity of a geometric series when $|r| < 1$. It does *not* require formal limit notation for arbitrary sequences. In this chapter, recurrence questions are therefore solved by finite algebraic structure—cycles, substitutions, invariants and exact formulas—rather than by using formal limit expressions.

What to look for in a recurrence

- **Repeated state:** if the next term depends on one previous term, one repeated value proves a cycle; if it depends on two previous terms, repeat an adjacent pair before declaring periodicity.
- **Affine shift:** for $u_{n+1} = pu_n + q$, find the fixed point L satisfying $L = pL + q$ and set $v_n = u_n - L$.
- **Reciprocal substitution:** if u_n appears in a denominator, $v_n = 1/u_n$ often converts the recurrence into an AP.
- **Fractional-linear recurrence:** fixed points frequently suggest a ratio such as $(u_n - r)/(u_n - s)$.
- **Second-order recurrence:** expressions involving two consecutive terms may be invariant even when the individual terms grow rapidly.
- **Integer recurrences:** parity, divisibility and cycles modulo a small integer can reveal the requested late term without computing everything.



A recurrence can be understood dynamically: each iterate is obtained by feeding the previous output back into the rule.

WORKED EXAMPLE 1 Explicit formula plus an integer threshold

The sequence is

$$u_n = 3n^2 - 14n + 11.$$

Find the least positive integer n for which $u_n > 150$.

SOLUTION

We need

$$3n^2 - 14n + 11 > 150 \iff 3n^2 - 14n - 139 > 0.$$

The positive root of $3n^2 - 14n - 139 = 0$ is

$$\frac{14 + \sqrt{196 + 1668}}{6} = \frac{14 + 4\sqrt{116.5}}{6} \approx 9.53.$$

Since the quadratic opens upward, the inequality holds for integers $n \geq 10$. Hence

$$\boxed{n = 10}.$$

COMMON TRAP. The question asks for an integer n . Do not stop at the real root of the quadratic.

WORKED EXAMPLE 2 A recurrence whose period only appears after enough terms

Let

$$x_0 = 1, \quad x_1 = 3, \quad x_{n+2} = |x_{n+1} - x_n|.$$

Find x_{100} and $\sum_{n=0}^{100} x_n$.

SOLUTION

Write enough terms to repeat an adjacent pair:

$$1, 3, 2, 1, 1, 0, 1, 1, 0, 1, 1, 0, \dots$$

From x_4 onward the block 1, 0, 1 repeats, equivalently x_n has period 3 from $n = 3$ onward with block 1, 1, 0. Since $100 = 3 + 97$ and $97 \equiv 1 \pmod{3}$, x_{100} is the second term of the repeating block, so

$$\boxed{x_{100} = 1}.$$

For the sum, the first three terms give $1 + 3 + 2 = 6$. The remaining indices 3 to 100 contain 98 terms: 32 full blocks of sum 2, followed by the first two terms 1, 1. Thus

$$\sum_{n=0}^{100} x_n = 6 + 32(2) + 2 = \boxed{72}.$$

KEY IDEA. With a second-order recurrence, repeating one value is not enough. A repeated adjacent pair determines all future terms.

WORKED EXAMPLE 3 Affine recurrence solved by shifting to its fixed point

A sequence satisfies

$$a_1 = 7, \quad a_{n+1} = 3a_n - 8.$$

Find a_{12} exactly.

SOLUTION

The fixed point L satisfies

$$L = 3L - 8 \implies L = 4.$$

Set $b_n = a_n - 4$. Then

$$b_{n+1} = a_{n+1} - 4 = 3a_n - 12 = 3(a_n - 4) = 3b_n.$$

So (b_n) is geometric with $b_1 = 3$, hence

$$b_n = 3^n.$$

Therefore

$$a_n = 4 + 3^n,$$

and

$$\boxed{a_{12} = 4 + 3^{12} = 531445}.$$

WORKED EXAMPLE 4 Sum of an affine recurrence without listing the terms

For the sequence

$$a_1 = 5, \quad a_{n+1} = 2a_n + 6,$$

find $\sum_{n=1}^{15} a_n$.

SOLUTION

The fixed point is $L = -6$, since $L = 2L + 6$. Put $b_n = a_n + 6$. Then

$$b_{n+1} = 2b_n,$$

with $b_1 = 11$. Hence

$$a_n = 11 \cdot 2^{n-1} - 6.$$

Therefore

$$\sum_{n=1}^{15} a_n = 11 \sum_{n=1}^{15} 2^{n-1} - 6 \cdot 15 = 11(2^{15} - 1) - 90.$$

Thus

$$\boxed{\sum_{n=1}^{15} a_n = 360347}.$$

WORKED EXAMPLE 5 Reciprocal substitution turns a nonlinear recurrence into an AP

Let

$$a_1 = 2, \quad a_{n+1} = \frac{a_n}{1 + 4a_n}.$$

Find a_{50} .**SOLUTION**

Take reciprocals:

$$\frac{1}{a_{n+1}} = \frac{1 + 4a_n}{a_n} = \frac{1}{a_n} + 4.$$

Thus $b_n = 1/a_n$ is an AP with $b_1 = 1/2$ and common difference 4:

$$b_n = \frac{1}{2} + 4(n-1) = 4n - \frac{7}{2} = \frac{8n-7}{2}.$$

Hence

$$\boxed{a_n = \frac{2}{8n-7}}, \quad \boxed{a_{50} = \frac{2}{393}}.$$

WORKED EXAMPLE 6 The same recurrence creates a finite telescoping sum

For the sequence in the previous example, evaluate

$$\sum_{n=1}^{50} a_n a_{n+1}.$$

SOLUTION

We have

$$a_n = \frac{2}{8n-7}, \quad a_{n+1} = \frac{2}{8n+1}.$$

Therefore

$$a_n a_{n+1} = \frac{4}{(8n-7)(8n+1)}.$$

Now

$$\frac{1}{8n-7} - \frac{1}{8n+1} = \frac{8}{(8n-7)(8n+1)},$$

so

$$a_n a_{n+1} = \frac{1}{2} \left(\frac{1}{8n-7} - \frac{1}{8n+1} \right).$$

Writing the first few terms makes the cancellation visible:

$$\frac{1}{2} \left(1 - \frac{1}{9} \right) + \frac{1}{2} \left(\frac{1}{9} - \frac{1}{17} \right) + \cdots + \frac{1}{2} \left(\frac{1}{393} - \frac{1}{401} \right).$$

Everything in the middle cancels. Hence

$$\sum_{n=1}^{50} a_n a_{n+1} = \frac{1}{2} \left(1 - \frac{1}{401} \right) = \boxed{\frac{200}{401}}.$$

KEY IDEA. Telescoping is a finite cancellation technique here. There is no need to introduce limits of sequences.

WORKED EXAMPLE 7 Fractional-linear recurrence reduced by its fixed points

Let

$$a_1 = 5, \quad a_{n+1} = \frac{3a_n + 1}{a_n + 3}.$$

Find a simple formula for

$$b_n = \frac{a_n - 1}{a_n + 1},$$

and hence find a_{10} .

SOLUTION

Compute directly:

$$b_{n+1} = \frac{\frac{3a_n+1}{a_n+3} - 1}{\frac{3a_n+1}{a_n+3} + 1} = \frac{2a_n - 2}{4a_n + 4} = \frac{1}{2} \frac{a_n - 1}{a_n + 1} = \frac{1}{2} b_n.$$

Thus (b_n) is a GP. Since

$$b_1 = \frac{5 - 1}{5 + 1} = \frac{2}{3},$$

we have

$$b_n = \frac{2}{3} \left(\frac{1}{2}\right)^{n-1} = \frac{2^{2-n}}{3}.$$

For $n = 10$, $b_{10} = 1/768$. Now solve

$$\frac{a_{10} - 1}{a_{10} + 1} = \frac{1}{768}.$$

So

$$768a_{10} - 768 = a_{10} + 1 \implies 767a_{10} = 769,$$

therefore

$$a_{10} = \frac{769}{767}.$$

WORKED EXAMPLE 8 An invariant in a second-order recurrence

Suppose

$$u_1 = 1, \quad u_2 = 4, \quad u_{n+2} = 4u_{n+1} - u_n.$$

Show that

$$I_n = u_n^2 + u_{n+1}^2 - 4u_n u_{n+1}$$

is constant, and find its value.

SOLUTION

Using $u_{n+2} = 4u_{n+1} - u_n$,

$$I_{n+1} = u_{n+1}^2 + u_{n+2}^2 - 4u_{n+1}u_{n+2}.$$

Substitute:

$$I_{n+1} = u_{n+1}^2 + (4u_{n+1} - u_n)^2 - 4u_{n+1}(4u_{n+1} - u_n).$$

Expanding and collecting gives

$$I_{n+1} = u_n^2 + u_{n+1}^2 - 4u_n u_{n+1} = I_n.$$

Hence the expression is invariant. At $n = 1$,

$$I_1 = 1 + 16 - 16 = 1.$$