

# CF MATHS

TMUA PAPER 1

## TRIGONOMETRY

Teacher Notes • Expanded Worked-Example Edition

**SPECIFICATION NOTE** This edition is aligned to the current publicly available UAT-UK TMUA MM4.1–MM4.6 content: sine and cosine rules, triangle area, the ambiguous case, 2-D and 3-D trigonometry, radians, arc/sector/segment geometry, exact values, graphs and periodicity of sine/cosine/tangent, the identities  $\tan x = \sin x / \cos x$  and  $\sin^2 x + \cos^2 x = 1$ , and trigonometric equations on stated intervals. The mixed examples use other separately specified TMUA topics but do not require differentiation or integration of trigonometric functions.

**DESIGN OF THIS EDITION** Worked-example headings contain no difficulty labels. The mathematics becomes progressively more demanding. Graphs and diagrams appear before or beside many solutions so students can see the geometry, periodicity and root-count structure rather than treating trigonometry as symbol manipulation. The final bank is written entirely in TMUA-style multiple-choice form.

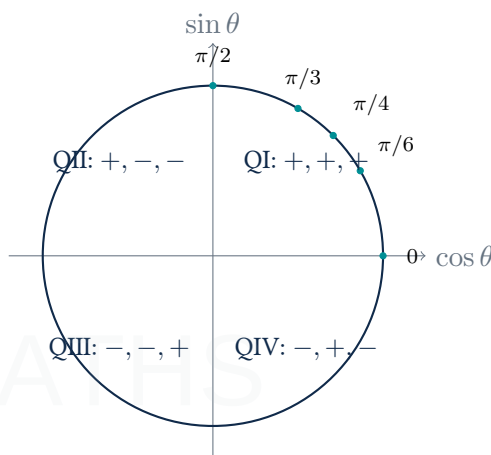
# 1. Exact Values, Quadrants and the Two Core Identities

The most valuable no-calculator habit is to separate a trigonometric question into three decisions: **reference angle**, **quadrant sign**, and **exact value**. A student should not memorise a separate table for every quadrant.

**Exact values to know instantly**

| $\theta$      | 0 | $\pi/6$      | $\pi/4$      | $\pi/3$      | $\pi/2$   |
|---------------|---|--------------|--------------|--------------|-----------|
| $\sin \theta$ | 0 | $1/2$        | $\sqrt{2}/2$ | $\sqrt{3}/2$ | 1         |
| $\cos \theta$ | 1 | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$        | 0         |
| $\tan \theta$ | 0 | $1/\sqrt{3}$ | 1            | $\sqrt{3}$   | undefined |

**CORE IDENTITIES.**  $\sin^2 x + \cos^2 x = 1$  and  $\tan x = \frac{\sin x}{\cos x}$  (when  $\cos x \neq 0$ ). These are the only identities that need to be treated as core tools here; do not invent compound-angle or double-angle methods when a simpler syllabus route exists.



Signs are read from the coordinates  $(\cos \theta, \sin \theta)$ ; tangent has the sign of their quotient.

## WORKED EXAMPLE 1 Quadrant information before calculation

Given  $90^\circ < \theta < 180^\circ$  and  $\cos \theta = -\sqrt{3}/2$ , find  $\sin \theta + \tan \theta$ .

### SOLUTION

Quadrant II gives  $\sin \theta > 0$  and  $\tan \theta < 0$ . The reference angle is  $30^\circ$ , so

$$\sin \theta = \frac{1}{2}, \quad \tan \theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Hence

$$\sin \theta + \tan \theta = \frac{1}{2} - \frac{\sqrt{3}}{3}.$$

**COMMON TRAP.** Determine signs before substituting exact values. A negative cosine does not force a negative sine.

**WORKED EXAMPLE 2** All solutions from one exact value

Solve  $\sin x = \frac{1}{2}$  for  $0 \leq x < 2\pi$ .

**SOLUTION**

The reference angle is  $\pi/6$ . Sine is positive in quadrants I and II, so

$$x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

**KEY IDEA.** One exact value can correspond to two, one or zero angles in an interval. Count using the graph/quadrants, not memory alone.

**WORKED EXAMPLE 3** Tangent and its shorter period

Solve  $\tan x = -1/\sqrt{3}$  for  $-\pi < x < \pi$ .

**SOLUTION**

The reference angle is  $\pi/6$ . Tangent is negative in quadrants II and IV and has period  $\pi$ . In the stated interval,

$$x = -\frac{\pi}{6}, \frac{5\pi}{6}.$$

**TMUA SHORTCUT.** Once one tangent solution is known, add or subtract  $\pi$  rather than  $2\pi$ .

**WORKED EXAMPLE 4** Recovering a hidden product

It is known that  $\sin \theta + \cos \theta = \frac{1}{2}$ . Find  $|\sin \theta - \cos \theta|$ .

**SOLUTION**

Square the given relation:

$$1 + 2 \sin \theta \cos \theta = \frac{1}{4} \implies \sin \theta \cos \theta = -\frac{3}{8}.$$

Then

$$(\sin \theta - \cos \theta)^2 = 1 - 2 \sin \theta \cos \theta = 1 + \frac{3}{4} = \frac{7}{4}.$$

Therefore

$$|\sin \theta - \cos \theta| = \frac{\sqrt{7}}{2}.$$

**WORKED EXAMPLE 5** A cubic in  $\sin x$  with root counting

How many solutions does

$$4 \sin^3 x - 4 \sin^2 x - \sin x + 1 = 0$$

have for  $0 \leq x < 2\pi$ ?

**SOLUTION**

Let  $s = \sin x$  with  $-1 \leq s \leq 1$ . Factor:

$$4s^3 - 4s^2 - s + 1 = (s - 1)(2s - 1)(2s + 1).$$

Thus  $s = 1, \frac{1}{2}, -\frac{1}{2}$ . On one full period, these give respectively 1, 2, 2 solutions. Hence

$$\boxed{5 \text{ solutions}}.$$

**TMUA SHORTCUT.** Solve the algebraic equation in  $s$  first; only then convert each admissible  $s$ -value into angles.

**WORKED EXAMPLE 6** Maximum and minimum on the restricted sine domain

Find the maximum and minimum values of

$$f(x) = 3 + 4 \sin x - 2 \sin^2 x.$$

**SOLUTION**

Set  $s = \sin x$ , so  $-1 \leq s \leq 1$ . Then

$$f = 3 + 4s - 2s^2 = 5 - 2(s - 1)^2.$$

The maximum is 5 at  $s = 1$ . Because the quadratic opens downward, the minimum on the restricted interval occurs at the endpoint farthest from 1, namely  $s = -1$ :

$$f_{\min} = 3 - 4 - 2 = -3.$$

Thus

$$\boxed{-3 \leq f(x) \leq 5}.$$

**COMMON TRAP.** The domain is not all real  $s$ ; it is  $[-1, 1]$ . Always check the endpoints.

**WORKED EXAMPLE 7** Absolute value and tangent root counting

How many solutions does

$$|\tan x| + \frac{2}{\tan x} = 4$$

have for  $0 \leq x < 2\pi$ ?

**SOLUTION**

Put  $t = \tan x \neq 0$ . If  $t > 0$ ,

$$t + \frac{2}{t} = 4 \iff t^2 - 4t + 2 = 0,$$

which has two positive roots. If  $t < 0$ ,

$$-t + \frac{2}{t} = 4 \iff t^2 + 4t - 2 = 0,$$

which has exactly one negative root. So there are three admissible tangent values. Each occurs twice on  $[0, 2\pi)$ , giving

6 solutions.

### WORKED EXAMPLE 8 A symmetric expression from a tangent value

Suppose  $\tan \theta = 2$  and  $\cos \theta < 0$ . Find  $\sin \theta - \cos \theta$  exactly.

#### SOLUTION

Since  $\tan \theta > 0$  and  $\cos \theta < 0$ ,  $\theta$  lies in quadrant III, so both sine and cosine are negative. Use a reference triangle with opposite 2, adjacent 1, hypotenuse  $\sqrt{5}$ :

$$\sin \theta = -\frac{2}{\sqrt{5}}, \quad \cos \theta = -\frac{1}{\sqrt{5}}.$$

Hence

$$\sin \theta - \cos \theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}.$$

### WORKED EXAMPLE 9 An integer parameter controlled by graph multiplicity

For how many integers  $k$  does

$$\cos x = \frac{k}{4}$$

have exactly two solutions on  $0 \leq x < 2\pi$ ?

#### SOLUTION

For exactly two solutions we need the horizontal level strictly between  $-1$  and  $1$ , but not at the endpoints  $\pm 1$  where there is one solution on  $[0, 2\pi)$ . Thus

$$-1 < \frac{k}{4} < 1 \iff -4 < k < 4.$$

The integers are  $-3, -2, -1, 0, 1, 2, 3$ , so

7.

### WORKED EXAMPLE 10 Exact values inside a rational expression

Evaluate exactly

$$\frac{\sin 30^\circ + \cos 60^\circ + \tan 45^\circ}{\cos 0^\circ + \sin 90^\circ}.$$

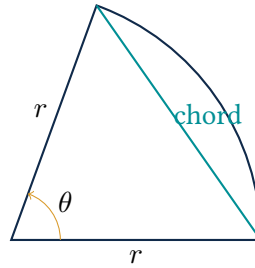
#### SOLUTION

The numerator is  $1/2 + 1/2 + 1 = 2$  and the denominator is  $1 + 1 = 2$ . Therefore

1.

## 2. Radians, Arcs, Sectors and Segments

Radians are not an optional alternative notation. The formulas  $s = r\theta$  and  $A = \frac{1}{2}r^2\theta$  require  $\theta$  in radians. Segment problems combine a sector with an isosceles triangle.



A minor segment equals sector area minus the area of triangle  $OAB$ .

### Core formulas

$$180^\circ = \pi \text{ radians}, \quad s = r\theta, \quad A_{\text{sector}} = \frac{1}{2}r^2\theta.$$

For a segment, find the included angle  $\theta$  and subtract  $\frac{1}{2}r^2 \sin \theta$  from the sector area when the triangle area is available from the ordinary area formula.

### WORKED EXAMPLE 11 Degrees to radians

Convert  $225^\circ$  to radians.

#### SOLUTION

$$225^\circ \cdot \frac{\pi}{180^\circ} = \boxed{\frac{5\pi}{4}}.$$

### WORKED EXAMPLE 12 Arc length determines the angle

A circle has radius 12 and an arc of length  $5\pi$ . Find the angle subtended at the centre.

#### SOLUTION

Using  $s = r\theta$ ,

$$5\pi = 12\theta \quad \Rightarrow \quad \boxed{\theta = \frac{5\pi}{12}}.$$